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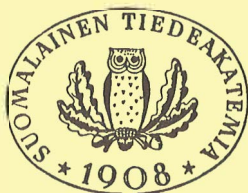
I. MATHEMATICA

DISSERTATIONES

57

HOMOGENEOUS AND CONFORMALLY
INVARIANT VARIATIONAL INTEGRALS

TERO KILPELÄINEN



HELSINKI 1985
SUOMALAINEN TIEDEAKATEMIA

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Jyväskylä, September 1985

TERO KILPELÄINEN

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1. Introduction

We consider variational integrals of the form

$$I(u) = \int_G F(x, \nabla u(x)) \, dx ,$$

where G is an open set in $\mathbb{R}^n$, u is in Sobolev space $W_p^1(G)$, $1 \leq p < \infty$ and $F(x, h) \approx |h|^p$. Usually the kernels F are assumed to satisfy certain natural conditions of measurability, convexity and growth, see e.g. [GLM, M, R3]. In Section 3 we shall determine the structure of the kernels which are assumed to be homogeneous in addition. The fundamental example $|h|^p$ of a kernel of this type turns out to be typical, i.e., $h \mapsto F(x, h)^{1/p}$ is always a norm. This fact will be used in Section 4, where we introduce a new norm in $L^p(G)^n$,

$$\|u\|_F = \left(\int_G F(x, u(x)) \, dx \right)^{1/p} ,$$

called an F -norm. The concept of an F -norm leads to a simple proof for the lower semicontinuity of the variational integral I with respect to weak convergence, a fundamental tool in Calculus of Variations introduced already by L. Tonelli [T], see also [L, M, S, R1, R3]. In weak topology the lower semicontinuity of a norm is an elementary fact and it gives a proof for the lower semicontinuity of a variational integral in our case. At the end of Section 4 we shall characterize the kernels F whose induced norm makes $L^p(G)^n$ uniformly convex.

The rest of this paper deals with the conformal invariance of variational integrals. Section 5 paves the way for the general case in Section 6. It is well-known that the conformal capacity is conformally invariant. We show in Section 5 that its natural generalization, the F -capacity, is not necessarily conformally invariant.

In the last section we study the conformal invariance of the variational integral

$$I(u) = \int_G F(x, u(x), \nabla u(x)) \, dx ,$$

where the kernel F satisfies only the familiar measurability conditions of Carathéodory and the natural growth restrictions. The well-known example is the n -Dirichlet integral

$$\int_G |\nabla u|^n \, dm .$$

We will show that it is a good prototype for all conformally invariant variational integrals, since every such integral in $\mathbb{R}^n$ is of the form

$$I(u) = \int_G k(x, u(x)) |\nabla u(x)|^n \, dx .$$

M. Grüter [Gr] has obtained the same result in the plane case under additional assumptions on the regularity of F . He also makes use of the fact that F does not depend on x . Our proof uses the norm characterization of the kernels (Section 3) as a model and is based on the fact that only the euclidean balls remain invariant under all orthogonal maps.

2. Preliminaries

2.1. *Notation.* For each set A in the euclidean space $\mathbb{R}^n$, $n \geq 2$, we let $\bar{A}$ and ∂A denote the closure and boundary of A , both taken with respect to $\mathbb{R}^n$. Given two sets $A \subset B$ in $\mathbb{R}^n$, $A \subset\subset B$ means that $\bar{A}$ is compact in B . By $x \cdot y$ we denote the usual inner product of two vectors x and y in $\mathbb{R}^n$, the euclidean norm of x is $|x| = (x \cdot x)^{1/2}$. For $x \in \mathbb{R}^n$ we use representations $x = (x_1, x_2, \dots, x_n) = \sum_{i=1}^n x_i e_i$. If $x \in \mathbb{R}^n$ and $r > 0$, we let $B(x, r)$ denote the open ball $\{y \in \mathbb{R}^n \mid |y - x| < r\}$, and $S^{n-1}(x, r)$ is the sphere $\partial B(x, r)$. Furthermore, we let $Q(x, r)$ denote the open cube $\{y \in \mathbb{R}^n \mid |y_i - x_i| < r, i = 1, 2, \dots, n\}$. We shall employ the abbreviations $B(r) = B(0, r)$, $S^{n-1}(r) = S^{n-1}(0, r)$, $S^{n-1} = S^{n-1}(1)$, $Q(r) = Q(0, r)$ and $Q_{n-1}(r) = Q(r) \cap \{x \in \mathbb{R}^n \mid x_1 = 0\}$.

We let $GL(\mathbb{R}^n, \mathbb{R}^n)$ denote the space of all regular linear maps $A: \mathbb{R}^n \rightarrow \mathbb{R}^n$ with the sup-norm.

The Lebesgue measure of a set $A \subset \mathbb{R}^n$ will be written as $m_n(A) = m(A)$.

If $A \subset \mathbb{R}^n$, then $C(A)$ is the class of all continuous (real valued) functions on A . If U is an open set in $\mathbb{R}^n$, we let $C^k(U)$ denote the family of all k times continuously differentiable functions $u: U \rightarrow \mathbb{R}$, and $C_0^k(U)$ the family of all $u \in C^k(U)$ whose support $\text{spt } u$ is a compact subset of U .

2.2. *Sobolev space W_p^1 and ACL^p -functions.* If A is a Lebesgue measurable subset of $\mathbb{R}^n$, then $L^p(A)$, $p \geq 1$, is the Banach space of all measurable functions $u: A \rightarrow \mathbb{R} = \mathbb{R} \cup \{-\infty, \infty\}$ with the norm

$$\|u\|_p = \|u\|_{p,A} = \left(\int_A |u|^p \, dm \right)^{1/p}.$$

Given an open set G in $\mathbb{R}^n$, $W_p^1(G)$ is the Sobolev space of functions $u \in L^p(G)$ whose first distributional partial derivatives $D_i u$ belong to $L^p(G)$ with the norm

$$\|u\|_{1,p} = \|u\|_p + \sum_{i=1}^n \|D_i u\|_p.$$

We let $\nabla u = (D_1 u, D_2 u, \dots, D_n u)$ denote the gradient of u . The space $W_{p,0}^1(G)$ is the closure of $C_0^\infty(G)$ in $W_p^1(G)$.

A mapping $f: G \rightarrow \mathbb{R}^m$ is said to be ACL if f is continuous and if for each open n -interval $Q \subset\subset G$, f is absolutely continuous on almost every line segment in $\overline{Q}$ parallel to the coordinate axes. An ACL -mapping has partial derivatives a.e. If these are locally L^p -integrable, $p \geq 1$, f is said to be ACL^p . It is well-known that f is ACL^p if and only if f is continuous and belongs to $W_p^1(D)$ for each open set $D \subset\subset G$. For basic properties of Sobolev spaces and ACL -functions see [M].

2.3. *C-functions.* Let G be an open set in $\mathbb{R}^n$. Functions $F: G \times \mathbb{R}^m \rightarrow \mathbb{R}$ satisfying the Carathéodory conditions

$$\begin{aligned} x \mapsto F(x, h) & \text{ is measurable for all } h \in \mathbb{R}^m, \\ h \mapsto F(x, h) & \text{ is continuous for a.e. } x \in G, \end{aligned}$$

will be called *Carathéodory-functions*, abbreviated *C-functions*. The following *Scorza-Dragoni property*, see [ET, p. 235], is well-known.

2.4. Lemma. A function $F: G \times \mathbb{R}^m \rightarrow \mathbb{R}$ is a C-function if and only if for each open set $D \subset\subset G$ and $\varepsilon > 0$ there is a compact set $C \subset D$ such that $m(D \setminus C) < \varepsilon$ and $F|_{C \times \mathbb{R}^m}$ is continuous.

2.5. Weak convergence. If X is a topological vector space and if X' stands for its dual, then a sequence $x_n \in X$, $n = 1, 2, \dots$, is said to *converge weakly* to $x_0 \in X$, abbreviated $x_n \rightarrow x_0$ weakly (in X), if

$$\lim_{n \rightarrow \infty} f(x_n) = f(x_0)$$

for all $f \in X'$.

In particular, $u_n \rightarrow u$ weakly in $L^p(A)$ means that

$$\lim_{n \rightarrow \infty} \int_A u_n v \, dm = \int_A u_0 v \, dm$$

for each $v \in L^q(A)$, where $q = p/(p - 1)$; if $p = 1$, then $q = \infty$.

The following elementary consequence of Hahn-Banach theorem is well-known, see e.g. [DS, Lemma II.3.27].

2.6. Lemma. Let X be a normed space and $x_n \in X$, $n = 0; 1, 2, \dots$. If $x_n \rightarrow x_0$ weakly, then

$$\|x_0\| \leq \liminf_{n \rightarrow \infty} \|x_n\|.$$

2.7. Conformal mappings. Let G and G' be domains in $\mathbb{R}^n$. A homeomorphism $f: G \rightarrow G'$ is *conformal* if $f \in C^1(G)$ and if

$$|f'(x) h| = |f'(x)| |h|$$

for every $x \in G$ and $h \in \mathbb{R}^n$. Here $|A|$ denotes the sup-norm of a linear map A . Alternatively, a C^1 -homeomorphism f is conformal if and only if $|f'(x)|^n = |J(x, f)|$ for all $x \in G$, where $J(x, f)$ denotes the Jacobian determinant of f at x . We shall frequently use

the following fact: Suppose that u is an ACL^p -function in G' and that $f: G \rightarrow G'$ is a C^1 -mapping. Then $v = u \circ f$ is ACL^p in G and

$$\nabla v(x) = f'(x)^* \nabla u(f(x))$$

a.e. in G ; see e.g. [GLM, 6.10]. As usual, A^* is the adjoint of a linear map $A: \mathbb{R}^n \rightarrow \mathbb{R}^n$.

A function $f: G \rightarrow \mathbb{R}^n$ is a *similarity* if $f(x) = \lambda O(x) + h$, $x \in G$, for some orthogonal map $O: \mathbb{R}^n \rightarrow \mathbb{R}^n$ and for some $\lambda \in \mathbb{R}$, $\lambda \neq 0$, and $h \in \mathbb{R}^n$.

3. Variational kernels

We are mainly interested in variational integrals

$$I(u) = \int F(x, \nabla u(x)) \, dx,$$

where $F(x, h) \approx |h|^p$. Integrals of this type have been extensively studied, see e.g. [GLM, M, R3]. In this section we shall derive some alternative characterizations for kernels F of the elliptic and homogeneous type.

Suppose that G is an open set in $\mathbb{R}^n$ and that the kernel $F: G \times \mathbb{R}^n \rightarrow \mathbb{R}$ satisfies the following assumptions, cf. [GLM]:

- (a) For each open set $D \subset\subset G$ and $\varepsilon > 0$ there is a compact set $C \subset D$ with $m(D \setminus C) < \varepsilon$ and $F|_{C \times \mathbb{R}^n}$ is continuous.
- (b) For a.e. $x \in G$ the function $h \rightarrow F(x, h)$ is convex.
- (c) There are constants $1 \leq p < \infty$ and $0 < \alpha \leq \beta < \infty$ such that for a.e. $x \in G$

$$\alpha |h|^p \leq F(x, h) \leq \beta |h|^p$$

for all $h \in \mathbb{R}^n$.

- (d) For a.e. $x \in G$, $F(x, \lambda h) = |\lambda|^p F(x, h)$ for all $\lambda \in \mathbb{R}$ and $h \in \mathbb{R}^n$.

The kernels F can also be characterized as follows.

3.1. Theorem. A kernel $F: G \times \mathbb{R}^n \rightarrow \mathbb{R}$ satisfies (a)-(d) if and only if it satisfies the following two conditions:

(i) For all $h \in \mathbb{R}^n$ the function $x \mapsto F(x, h)$ is measurable in G .

(ii) There exist a constant $p \in [1, \infty)$ and a set $D \subset G$ such that $m(G \setminus D) = 0$ and

$$P = \{ h \mapsto F(x, h)^{1/p} \mid x \in D \}$$

is a uniformly equivalent family of norms in $\mathbb{R}^n$.

A family P of norms $f: \mathbb{R}^n \rightarrow \mathbb{R}$ is called *uniformly equivalent* if there exist constants $0 < \alpha \leq \beta < \infty$ such that $\alpha |h| \leq f(h) \leq \beta |h|$ for all $h \in \mathbb{R}^n$ and $f \in P$.

Theorem 3.1 is an immediate consequence of the Scorza-Dragoni property (2.4) and the following lemma.

3.2. Lemma. Suppose that X is a vector space and that $p \in [1, \infty)$. A non-negative function $f: X \rightarrow \mathbb{R}$ is a norm if and only if

- (i) f^p is convex,
- (ii) $f(\lambda x)^p = |\lambda|^p f(x)^p$ for all $\lambda \in \mathbb{R}$ and $x \in X$,
- (iii) $f(x)^p = 0$ implies $x = 0$.

Proof. Obviously, a norm $f: X \rightarrow \mathbb{R}$ satisfies (i)-(iii). For the converse, it suffices to show the triangle inequality. Fix $x, y \in X$. Set $f(x) = t$ and $f(y) = s$. We may assume that $ts \neq 0$. Write $u = t + s$. Now

$$\frac{x+y}{u} = \frac{t}{u} \left(\frac{x}{t} \right) + \frac{s}{u} \left(\frac{y}{s} \right),$$

whence the convexity of f^p together with (ii) yield

$$f\left(\frac{x+y}{u}\right)^p \leq \frac{t}{u} f\left(\frac{x}{t}\right)^p + \frac{s}{u} f\left(\frac{y}{s}\right)^p = \frac{t+s}{u} = 1.$$

Thus the condition (ii) implies the triangle inequality.

3.3. Remark. A similar reasoning to above gives: For a fixed $p \in [1, \infty)$ a non-negative function $f: X \rightarrow \mathbb{R}$ is a seminorm if and only

if the conditions (i) and (ii) of Lemma 3.2 are satisfied.

A norm $f: X \rightarrow \mathbb{R}$ is called *strict* if $f(x+y) = f(x) + f(y)$ implies $y = 0$ or $x = \lambda y$ for some $\lambda \geq 0$.

A kernel function $F: G \times \mathbb{R}^n \rightarrow \mathbb{R}$ is said to be *strictly convex* if for a.e. $x \in G$

$$F(x, ty + (1-t)h) < tF(x, y) + (1-t)F(x, h)$$

for $t \in (0,1)$ and $y, h \in \mathbb{R}^n$, $y \neq h$.

3.4. Theorem. Suppose that $F: G \times \mathbb{R}^n \rightarrow \mathbb{R}$ satisfies (a)-(d) for some $1 < p < \infty$. Then F is strictly convex if and only if for a.e. $x \in G$ the norm $h \mapsto F(x, h)^{1/p}$ is strict.

Proof. The sufficiency easily follows from the strict convexity of $h \mapsto |h|^p$. For the converse, let $f(h) = F(x, h)^{1/p}$ be a norm such that f^p is strictly convex. Fix $h, y \in \mathbb{R}^n$ with $f(h+y) = f(h) + f(y)$. We may assume that $h, y \neq 0$. As in the proof for Lemma 3.2 we obtain

$$1 = f\left(\frac{h+y}{u}\right)^p \leq \frac{t}{u} f\left(\frac{h}{t}\right)^p + \frac{s}{u} f\left(\frac{y}{s}\right)^p = 1,$$

where $f(h) = t$, $f(y) = s$ and $u = t + s$. Observe that $t, s \in (0, u)$. Thus the strict convexity of f^p yields

$$\frac{h}{t} = \frac{y}{s}.$$

This completes the proof.

3.5. Inner product kernels. If $\langle \cdot, \cdot \rangle: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ is an inner product in $\mathbb{R}^n$, there is a self-adjoint, positively definite linear mapping $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$ with

$$(3.6) \quad \langle h, y \rangle = Th \cdot y$$

for all $h, y \in \mathbb{R}^n$, and, conversely, by (3.6) each such mapping T

induces an inner product in $\mathbb{R}^n$. Thus a norm f in $\mathbb{R}^n$ is induced by an inner product if and only if there exists a self-adjoint T such that

$$f(h) = (Th \cdot h)^{1/2}$$

for all $h \in \mathbb{R}^n$.

If $\theta: G \rightarrow GL(\mathbb{R}^n, \mathbb{R}^n)$ is a measurable mapping such that for a.e. $x \in G$ the linear map $\theta(x)$ is self-adjoint and that there exist constants $0 < a \leq b < \infty$ with

$$a|h|^2 \leq \theta(x) h \cdot h \leq b|h|^2$$

for a.e. $x \in G$ and all $h \in \mathbb{R}^n$, then the kernels

$$F_\theta(x, h) = (\theta(x) h \cdot h)^{p/2}, \quad 1 \leq p < \infty,$$

satisfy (a)-(d). Note that a kernel function F is of the type F_θ if and only if for a.e. $x \in G$ the norm $h \mapsto F(x, h)^{1/p}$ is induced by an inner product. Therefore we call kernels F_θ *inner product kernels*. These kernels have been studied by Yu. G. Reshetnyak [R2, R3] in the case $p = n$. They play an essential role in the theory of quasiregular mappings; see [R2, R3] and also [BI, GLM].

The inner product property is connected to the smoothness of F :

3.7. Theorem. *Let $F: G \times \mathbb{R}^n \rightarrow \mathbb{R}$ satisfy (a)-(d) for $p \in [1, \infty)$. Then F is an inner product kernel if and only if for a.e. $x \in G$ the function $h \mapsto F(x, h)^{2/p}$ belongs to $C^2(\mathbb{R}^n)$.*

Proof. For an inner product kernel F , the function $h \mapsto F(x, h)^{2/p}$ is a C^∞ -function in $\mathbb{R}^n$ for a.e. $x \in G$. For the converse, it suffices to prove: If $f(h) = F(x, h)^{1/p}$ is a norm such that $g = f^2 \in C^2(\mathbb{R}^n)$; then the parallelogram law

$$g(h + y) + g(h - y) = 2g(h) + 2g(y)$$

holds for all $h, y \in \mathbb{R}^n$; see [Y, p. 39].

First note that the homogeneity of f implies

$$\partial_i \partial_j g(\lambda h) = \partial_i \partial_j g(h) , \quad i, j = 1, 2, \dots, n ,$$

for all $\lambda \neq 0$ and $h \in \mathbb{R}^n$. Letting $\lambda \rightarrow 0$ we obtain

$$\partial_i \partial_j g(0) = \partial_i \partial_j g(h) , \quad i, j = 1, 2, \dots, n ,$$

for all $h \in \mathbb{R}^n$. Now Taylor's formula yields

$$(3.8) \quad g(h) = \frac{1}{2} \sum_{i,j=1}^n \partial_i \partial_j g(0) h_i h_j$$

for all $h \in \mathbb{R}^n$.

To complete the proof let $h, y \in \mathbb{R}^n$. By (3.8)

$$\begin{aligned} & g(h+y) + g(h-y) \\ &= \frac{1}{2} \sum_{i,j=1}^n \partial_i \partial_j g(0) (h_i + y_i)(h_j + y_j) + \frac{1}{2} \sum_{i,j=1}^n \partial_i \partial_j g(0) (h_i - y_i)(h_j - y_j) \\ &= \sum_{i,j=1}^n \partial_i \partial_j g(0) (h_i h_j + y_i y_j) = 2g(h) + 2g(y) \end{aligned}$$

as desired.

3.9. Remark. Theorem 3.7 yields for $p = 2$: If $h \mapsto F(x, h)$ belongs to $C^p(\mathbb{R}^n)$, then F is an inner product kernel. For $p \neq 2$ this is not true. For example, for $n \geq 2$, consider the kernels

$$F(x, h) = \sum_{i=1}^n |h_i|^p , \quad 1 \leq p < \infty .$$

Observe that the norm

$$f(h) = \left(\sum_{i=1}^n |h_i|^p \right)^{1/p}$$

is induced by an inner product if and only if $p = 2$. However, the function $h \mapsto F(x, h)$ belongs to $C^p(\mathbb{R}^n)$ for $p = 2, 3, 4, \dots$. Furthermore, $h \mapsto F(x, h)$ even belongs to $C^\infty(\mathbb{R}^n)$ for $p = 2k$, $k = 1, 2, \dots$.

4. F-norm

A variational kernel F satisfying (a)-(d) induces a new norm, called an F -norm in $L^p(G)^n$. This leads to a simple proof for the lower semicontinuity of a variational integral, and we also study the uniform convexity of $L^p(G)^n$, $p \in (1, \infty)$, under the F -norm.

4.1. *F-norm.* Suppose that G is an open set in $\mathbb{R}^n$ and that $F: G \times \mathbb{R}^n \rightarrow \mathbb{R}$ satisfies (a)-(d) for some $p \in [1, \infty)$. We denote by $L^p(G)^n$ the cartesian product $L^p(G) \times \dots \times L^p(G)$ (n times) with the norm

$$\|u\|_{L^p} = \left(\sum_{i=1}^n \|u_i\|_p^p \right)^{1/p},$$

where $u = (u_1, \dots, u_n) \in L^p(G)^n$ and $\|u_i\|_p$, $i = 1, 2, \dots, n$, is the usual $L^p(G)$ -norm of u_i . We introduce another norm $\|\cdot\|_F$ in $L^p(G)^n$, called an F -norm,

$$\|u\|_F = \left(\int_G F(x, u(x)) \, dx \right)^{1/p}.$$

The assumptions (a)-(d) together with Lemmas 2.4 and 3.2 immediately yield:

4.2. *Lemma.* F -norm is a norm in $L^p(G)^n$ equivalent to the norm $\|\cdot\|_{L^p}$.

4.3. *Remarks.* (1) If $p = 2$ and F is an inner product kernel, then $L^2(G)^n$ is a Hilbert space under the F -norm.

(2) If $A \subset G$ is a measurable set, we write

$$\|u\|_{F,A} = \left(\int_A F(x, u) \, dx \right)^{1/p}$$

for $u \in L^p(A)^n$. Obviously, $\|u\|_{F,A}$ is equivalent to the usual $L^p(A)^n$ -norm

$$\|u\|_{L^p,A} = \left(\sum_{i=1}^n \|u_i\|_{p,A}^p \right)^{1/p}.$$

4.4. *Lower semicontinuity.* The concept of an F-norm can be used to give a simple proof for the lower semicontinuity of the variational integral

$$I(u) = \int_G F(x, \nabla u(x)) \, dx$$

under the assumptions (a)-(d). There exist several proofs for this fundamental result in a more general case; see e.g. [D, M, R1, R3, S].

For a simple proof based on Banach-Saks' theorem see [L]. These proofs do not make use of the homogeneity assumption (d).

4.5. *Theorem.* Let I and G be as above and let u_i , $i = 0; 1, 2, \dots$, be a sequence of W_P^1 -functions in G such that $\nabla u_i \rightarrow \nabla u_0$ weakly in $L^P(G)^n$. Then

$$I(u_0) \leq \liminf_{i \rightarrow \infty} I(u_i) .$$

Proof. In $L^P(G)^n$ we use the F-norm. By Lemma 4.2, $\nabla u_i \rightarrow \nabla u_0$ weakly in $L^P(G)^n$. Thus Lemma 2.6 implies

$$\|\nabla u_0\|_F \leq \liminf_{i \rightarrow \infty} \|\nabla u_i\|_F ,$$

and hence

$$I(u_0) \leq \liminf_{i \rightarrow \infty} I(u_i)$$

as desired.

4.6. *Remarks.* (1) The above reasoning can also be used in a more general situation. Let $F: G \times \mathbb{R}^k \times \mathbb{R}^{nk} \rightarrow \mathbb{R}$ satisfy the conditions (a)-(d) modified in an obvious way (i.e., $F(x, \lambda u, \lambda v) = |\lambda|^P F(x, u, v)$). Then the following result holds. Let $u^{(i)} = (u_1^{(i)}, \dots, u_k^{(i)})$, $i = 0; 1, 2, \dots$, be a sequence in $W_P^1(G)^k$. If $\nabla u_j^{(i)} \rightarrow \nabla u_j^{(0)}$ and $u_j^{(i)} \rightarrow u_j^{(0)}$ weakly in $L^P(G)$, $j = 1, 2, \dots, k$, then

$$I(u^{(0)}) \leq \liminf_{i \rightarrow \infty} I(u^{(i)}) ,$$

where

$$I(u) = \int_G F(x, u_1(x), \dots, u_k(x), \nabla u_1(x), \dots, \nabla u_k(x)) \, dx .$$

(2) It is clear that in the above results we can take any measurable set $A \subset G$ instead of the open set G .

4.7. Corollary. Suppose that $G \subset \mathbb{R}^n$ is an open set and that $F: G \times \mathbb{R}^n \rightarrow \mathbb{R}$ satisfies (a)-(d). Furthermore, suppose that $F_i: G \times \mathbb{R}^n \rightarrow \mathbb{R}$, $i = 1, 2, \dots$, are such that

- (i) F_i satisfies (a) for each i .
- (ii) For a.e. $x \in G$, $F_i(x, h) \geq 0$ for all $h \in \mathbb{R}^n$ and $i \in \mathbb{N}$.
- (iii) For each $\varepsilon > 0$ there is a compact set $C \subset G$ such that $m(G \setminus C) < \varepsilon$ and $F_i \rightarrow F$ uniformly in $C \times \mathbb{R}^n$.

If u_m , $m = 0, 1, 2, \dots$, are W_p^1 -functions in G such that $\nabla u_m \rightarrow \nabla u_0$ weakly in $L^p(G)^n$, then

$$\lim_{i \rightarrow \infty} \int_G F_i(x, \nabla u_i(x)) \, dx \geq \int_G F(x, \nabla u_0(x)) \, dx .$$

Proof. Consider first a set $A \subset G$ with $m(A) < \infty$. Fix $\varepsilon > 0$ and choose compact sets $C_k \subset A$ with $m(A \setminus C_k) < \frac{1}{k}$ and $F_i \rightarrow F$ uniformly in $C_k \times \mathbb{R}^n$. Now there is an i_0 such that $i \geq i_0$ implies

$$F_i(x, h) \geq F(x, h) - \frac{\varepsilon}{m(A)}$$

for all $(x, h) \in C_k \times \mathbb{R}^n$. Theorem 4.5 (cf. 4.6 (2)) yields

$$\begin{aligned} \lim_{i \rightarrow \infty} \int_A F_i(x, \nabla u_i(x)) \, dx &\geq \lim_{i \rightarrow \infty} \int_{C_k} F_i(x, \nabla u_i(x)) \, dx \\ &\geq \lim_{i \rightarrow \infty} \int_{C_k} F(x, \nabla u_i(x)) \, dx - \frac{m(C_k)}{m(A)} \varepsilon \\ &\geq \int_{C_k} F(x, \nabla u_0(x)) \, dx - \frac{m(C_k)}{m(A)} \varepsilon . \end{aligned}$$

The last term tends to

$$\int_A F(x, \nabla u_0(x)) \, dx - \varepsilon$$

as $k \rightarrow \infty$. Letting $\varepsilon \rightarrow 0$ we obtain

$$\lim_{i \rightarrow \infty} \int_A F_i(x, \nabla u_i(x)) \, dx \geq \int_A F(x, \nabla u_0(x)) \, dx.$$

To complete the proof, note that the absolute continuity of the integral allows us to choose for each $\varepsilon > 0$ a compact set $A_\varepsilon \subset G$ such that

$$\int_G F(x, \nabla u_0(x)) \, dx \leq \int_{A_\varepsilon} F(x, \nabla u_0(x)) \, dx + \varepsilon.$$

Hence

$$\int_G F(x, \nabla u_0(x)) \, dx \leq \lim_{i \rightarrow \infty} \int_{A_\varepsilon} F_i(x, \nabla u_i(x)) \, dx + \varepsilon$$

$$\leq \lim_{i \rightarrow \infty} \int_G F_i(x, \nabla u_i(x)) \, dx + \varepsilon$$

for each $\varepsilon > 0$.

4.8. *Uniform convexity.* The rest of this section deals with uniform convexity; especially, we study the uniform convexity of $L^p(G)^n$, $p \in (1, \infty)$, under the F -norm.

A normed space $(X, \|\cdot\|)$ is called *uniformly convex* if for each $\varepsilon > 0$ there is $\delta > 0$ such that $\|x\| \leq 1$, $\|y\| \leq 1$ and $\|x - y\| \geq \varepsilon$ imply $\|\frac{1}{2}x + \frac{1}{2}y\| \leq 1 - \delta$. For alternative definitions see [K]. We use some of them in what follows.

We call a family N of norms $g: X \rightarrow \mathbb{R}$ *equiuniformly convex* if for each $\varepsilon > 0$ there is $\delta > 0$ such that for each $g \in N$ the conditions $g(x) \leq 1$, $g(y) \leq 1$ and $g(x - y) \geq \varepsilon$ imply

$$g(\frac{1}{2}x + \frac{1}{2}y) \leq 1 - \delta.$$

Further, if $U \subset X$ is a convex set and if F is a family of continuous functions $f: U \rightarrow \mathbb{R}$, then F is called *equistrictly convex* if for each compact set $C \subset U$ and $\varepsilon > 0$ there is $\delta > 0$ such that $x, y \in C$ and $\|x - y\| \geq \varepsilon$ imply

$$f\left(\frac{1}{2}x + \frac{1}{2}y\right) \leq (1 - \delta) \frac{f(x) + f(y)}{2}.$$

Obviously, an equiuniformly convex family of norms is not equistrictly convex if $\dim X \geq 1$. The connection between the above-mentioned concepts is discussed in 4.13; see also 4.22.

The following lemma is a generalization of McShane's idea [K, pp. 360-362, McS].

4.9. Lemma. Suppose that X is a normed space and that N is a family of norms in X . Let $p \in (1, \infty)$. Then N is equiuniformly convex if and only if for each $\varepsilon > 0$ there is $\delta > 0$ such that for every $f \in N$ the conditions $f(x) \leq 1$, $f(y) \leq 1$ and $f(x-y) \geq \varepsilon$ imply

$$(4.10) \quad f\left(\frac{1}{2}x + \frac{1}{2}y\right)^p \leq (1 - \delta) \frac{f(x)^p + f(y)^p}{2}.$$

Proof. The equiuniform convexity of N immediately follows from (4.10). To prove the converse, it suffices to show that for each $\varepsilon > 0$ there is $\delta > 0$ such that for every $f \in N$ the conditions $f(x) = 1$, $f(y) \leq 1$ and $f(x-y) \geq \varepsilon$ imply (4.10).

Let us suppose that there are $\varepsilon > 0$ and sequences $x_n, y_n \in X$ and $f_n \in N$ such that $f_n(x_n) = 1$, $f_n(y_n) \leq 1$, $f_n(x_n - y_n) \geq \varepsilon$ and that

$$(4.11) \quad \lim_{n \rightarrow \infty} \frac{f_n\left(\frac{1}{2}x_n + \frac{1}{2}y_n\right)^p}{\frac{1}{2} + \frac{1}{2}f_n(y_n)^p} = 1.$$

Then $\lim_{n \rightarrow \infty} f_n(y_n) = 1$, since otherwise there is $q < 1$ with $f_n(y_n) \leq q$ for some subsequence denoted again by $f_n(y_n)$. Then there exists $\rho < 1$ such that

$$(4.12) \quad f_n\left(\frac{1}{2}x_n + \frac{1}{2}y_n\right)^p \leq \left(\frac{1}{2}(1 + f_n(y_n))\right)^p \leq \frac{\rho}{2} (1 + f_n(y_n)^p),$$

since the function

$$t \mapsto \frac{(1+t)^p}{1+t^p}$$

is strictly increasing in $[0, 1]$; we may choose

$\rho = (1 + q)^p (1 + q^p)^{-1} 2^{1-p}$. Now (4.12) yields

$$\frac{f_n(\frac{1}{2}x_n + \frac{1}{2}y_n)^p}{\frac{1}{2} + \frac{1}{2}f_n(y_n)^p} \leq \rho < 1,$$

which contradicts (4.11). Hence $\lim_{n \rightarrow \infty} f_n(y_n) = 1$, and (4.11) implies

$$\lim_{n \rightarrow \infty} f_n(\frac{1}{2}x_n + \frac{1}{2}y_n) = 1,$$

which contradicts the equiuniform convexity of N . This completes the proof.

4.13. Corollary. Suppose that $p \in (1, \infty)$ and that F is a family of continuous functions $f: \mathbb{R}^n \rightarrow \mathbb{R}$ such that

- (i) $f(\lambda x) = |\lambda|^p f(x)$ for all $\lambda \in \mathbb{R}$, $x \in \mathbb{R}^n$, and
- (ii) there exist constants $0 < \alpha \leq \beta < \infty$ with

$$\alpha |x|^p \leq f(x) \leq \beta |x|^p$$

for all $x \in \mathbb{R}^n$ and $f \in F$.

Then F is equistrictly convex if and only if $F^{1/p} = \{f^{1/p} \mid f \in F\}$ is an equiuniformly convex family of norms.

Proof. Suppose that F is equistrictly convex. Then each $f \in F$ is strictly convex, cf. [RV, Section 71]. By Lemma 3.2, $F^{1/p}$ is a family of norms in $\mathbb{R}^n$. Choose

$$C = \overline{B(0, \alpha^{-1/p})}.$$

Then $f^{-1}([0, 1]) \subset C$ for each $f \in F$; hence Lemma 4.9 implies the equiuniform convexity of $F^{1/p}$. For the converse let $C \subset \mathbb{R}^n$ be a compact set, $C \neq \{0\}$. Then condition (ii) yields

$$0 < M = \sup \{g(x) \mid x \in C, g \in F^{1/p}\} < \infty.$$

Hence $\frac{1}{M} f^{1/p}(C) \subset [0, 1]$, and Lemma 4.9 implies the desired result.

The above connection between equistrictly convex and equiuniformly

convex families is not true in infinite dimensional spaces; see 4.22.

4.14. *Uniformly convex kernels.* Suppose that G is an open set in $\mathbb{R}^n$ and that $F: G \times \mathbb{R}^n \rightarrow \mathbb{R}$ satisfies (a)-(d) for $p \in (1, \infty)$. The kernel F is *uniformly convex* if there is $G' \subset G$ such that $m(G \setminus G') = 0$ and that the family $F = \{h \mapsto F(x, h) \mid x \in G'\}$ is equi-strictly convex.

Corollary 4.13 yields:

4.15. *Lemma.* Suppose that $p \in (1, \infty)$. The kernel $F: G \times \mathbb{R}^n \rightarrow \mathbb{R}$ is uniformly convex if and only if there is $G' \subset G$ such that $m(G \setminus G') = 0$ and $N = \{h \mapsto F(x, h)^{1/p} \mid x \in G'\}$ is an equiuniformly convex family of norms.

The next theorem says that all inner product kernels belong to this class.

4.16. *Theorem.* Suppose that G is an open set in $\mathbb{R}^n$ and that $F: G \times \mathbb{R}^n \rightarrow \mathbb{R}$ is an inner product kernel with $p \in (1, \infty)$. Then F is uniformly convex.

Proof. Fix $\epsilon > 0$ and choose

$$\delta = \begin{cases} 1 - (1 - \epsilon^2/4)^{1/2} & \text{if } \epsilon \leq 2 \\ 1 & \text{if } \epsilon \geq 2. \end{cases}$$

The parallelogram law yields that the desired inequality in 4.8 holds for an inner product norm $g(h) = F(x, h)^{1/p}$.

4.17. *Remark.* A strictly convex kernel is not necessarily uniformly convex, even though it is continuous. Fix $p \in (1, \infty)$ and let $G = B(0, 1) \setminus \{0\} \subset \mathbb{R}^n$. Define $F: G \times \mathbb{R}^n \rightarrow \mathbb{R}$ by

$$F(x, h) = \left(\sum_{i=1}^n |h_i|^{1/|x|} \right)^{|x|p}.$$

Then F is continuous and a strictly convex kernel, see 3.4. However,

F is not uniformly convex. To see this, consider sequences $h_i = (a_i, a_i, \dots, a_i)$ and $y_i = (-a_i, a_i, a_i, \dots, a_i)$, where $a_i = n^{-1/i}$, $i = 2, 3, \dots$. Choose x_i such that $|x_i| = i^{-1}$. Then $F(x_i, h_i) = 1 = F(x_i, y_i)$ and

$$\lim_{i \rightarrow \infty} F(x_i, \frac{1}{2}h_i + \frac{1}{2}y_i) = 1,$$

but

$$\lim_{i \rightarrow \infty} F(x_i, h_i - y_i) = 2^p > 0.$$

The next theorem characterizes the kernels F for which $L^p(G)^n$, $p \in (1, \infty)$, is uniformly convex under the F -norm. The sufficiency part generalizes an idea of McShane's [McS].

4.18. Theorem. *Suppose that G is an open set in $\mathbb{R}^n$ and that $F: G \times \mathbb{R}^n \rightarrow \mathbb{R}$ satisfies (a)-(d) for some $p \in (1, \infty)$. Then $L^p(G)^n$ is uniformly convex under the F -norm if and only if F is uniformly convex.*

Proof. Suppose first that F is uniformly convex. Fix $\varepsilon > 0$ and choose $f, g \in L^p(G)^n$ such that $\|f\|_F = 1$, $\|g\|_F = 1$ with $\|f - g\|_F \geq \varepsilon$. Write

$$A = \left\{ x \in G \mid F(x, f(x) - g(x)) > \frac{\varepsilon^p}{4} (F(x, f(x)) + F(x, g(x))) \right\}.$$

Then A is measurable, and by Lemmas 4.9 and 4.15 there is $\delta > 0$ such that

$$(4.19) \quad F(x, \frac{1}{2}f(x) + \frac{1}{2}g(x)) \leq (1 - \delta) \frac{F(x, f(x)) + F(x, g(x))}{2}$$

for a.e. $x \in A$. Since

$$\begin{aligned} & \int_{G \setminus A} F(x, f(x) - g(x)) \, dx \\ & \leq \frac{\varepsilon^p}{4} \int_G (F(x, f(x)) + F(x, g(x))) \, dx \leq \frac{\varepsilon^p}{2}, \end{aligned}$$

we obtain

$$\int_A F(x, f(x) - g(x)) \, dx \geq \epsilon^p - \frac{\epsilon^p}{2} = \frac{\epsilon^p}{2}.$$

Now $\|f - g\|_{F,A} \geq \epsilon 2^{-1/p}$, and hence $\max \{\|f\|_{F,A}^p, \|g\|_{F,A}^p\} \geq \epsilon^p 2^{-(p+1)}$. Thus the condition (b) and (4.19) yield

$$\begin{aligned} & \int_G \left(\frac{1}{2} F(x, f(x)) + \frac{1}{2} F(x, g(x)) - F(x, \frac{1}{2} f(x) + \frac{1}{2} g(x)) \right) dx \\ & \geq \int_A \left(\frac{1}{2} F(x, f(x)) + \frac{1}{2} F(x, g(x)) - F(x, \frac{1}{2} f(x) + \frac{1}{2} g(x)) \right) dx \\ & \geq \frac{\delta}{2} \int_A (F(x, f(x)) + F(x, g(x))) \, dx \geq \delta \epsilon^p 2^{-(p+2)}. \end{aligned}$$

Hence

$$\begin{aligned} \frac{1}{2} f + \frac{1}{2} g \|_F^p & \leq \frac{1}{2} \|f\|_F^p + \frac{1}{2} \|g\|_F^p - \delta \epsilon^p 2^{-(p+2)} \\ & \leq 1 - \delta \epsilon^p 2^{-(p+2)}, \end{aligned}$$

which implies the desired result.

To prove the converse, suppose that there exist $\epsilon > 0$, $A \subset G$ and sequences $u_k, v_k: G \rightarrow \mathbb{R}^n$, $k = 1, 2, \dots$, such that $m(A) > 0$ and for all $x \in A$, $F(x, u_k(x)) \leq 1$, $F(x, v_k(x)) \leq 1$, $F(x, u_k(x) - v_k(x)) \geq 2\epsilon$ and $F(x, \frac{1}{2}u_k(x) + \frac{1}{2}v_k(x)) > 1 - \frac{1}{k}$, cf. Lemma 4.15. By approximation we may suppose that $F(x, u_k(x)) < 1$ and $F(x, v_k(x)) < 1$ in A . Now the condition (a) gives a compact set $C \subset A$ such that $m(C) > 0$ and $F|_C \times \mathbb{R}^n$ is continuous. Fix $k \in \mathbb{N}$. The continuity of $F|_C \times \mathbb{R}^n$ allows us to choose for each $y \in C$ an open neighbourhood $N_{y,k}$ such that for every $x \in N_{y,k} \cap C$ $F(x, u_k(y)) < 1$, $F(x, v_k(y)) < 1$, $F(x, u_k(y) - v_k(y)) \geq \epsilon$ and $F(x, \frac{1}{2}u_k(y) + \frac{1}{2}v_k(y)) > 1 - \frac{2}{k}$. Since C is compact and $m(C) > 0$, there is $y_0 \in C$ with $m(N_{y_0,k} \cap C) > 0$. Write $h_k = u_k(y_0)$, $v_k = v_k(y_0)$ and $B_k = N_{y_0,k} \cap C$. Set

$$f_k(x) = m(B_k)^{-1/p} h_k \chi_{B_k}$$

and

$$g_k(x) = m(B_k)^{-1/p} y_k \chi_{B_k},$$

where χ_{B_k} stands for the characteristic function of B_k . Then f_k and g_k are measurable,

$$\int_G F(x, f_k(x)) dx < 1 \quad \text{and} \quad \int_G F(x, g_k(x)) dx < 1.$$

Thus $f_k, g_k \in L^p(G)^n$, $\|f_k\|_F \leq 1$, $\|g_k\|_F \leq 1$ and

$$\begin{aligned} \|f_k - g_k\|_F^p &= \int_G F(x, f_k(x) - g_k(x)) dx \\ &= m(B_k)^{-1} \int_{B_k} F(x, h_k - y_k) dx \geq \varepsilon. \end{aligned}$$

Furthermore,

$$\left\| \frac{1}{2} f_k + \frac{1}{2} g_k \right\|_F^p = m(B_k)^{-1} \int_{B_k} F(x, \frac{1}{2} h_k + \frac{1}{2} y_k) dx \geq 1 - \frac{2}{k},$$

which tends to 1 as $k \rightarrow \infty$. This contradicts the uniform convexity of $L^p(G)^n$ under the F-norm, and the proof is complete.

4.20. Corollary. Suppose that $F: G \times \mathbb{R}^n \rightarrow \mathbb{R}$ is a uniformly convex kernel. If $u_m \in W_p^1(G)$, $m = 0, 1, 2, \dots$ are such that $u_m \rightarrow u_0$ in $L^p(G)$ with

$$(4.21) \quad \overline{\lim}_{m \rightarrow \infty} \int_G F(x, \nabla u_m(x)) dx \leq \int_G F(x, \nabla u_0(x)) dx,$$

then $u_m \rightarrow u_0$ in $W_p^1(G)$.

Proof. We use the F-norm in $L^p(G)^n$. Since $L^p(G)^n$ is uniformly convex, it suffices to show that $\nabla u_m \rightarrow \nabla u_0$ weakly in $L^p(G)^n$; cf. Theorem 4.5 and [HS, p. 233]. This is easily seen by standard reasoning. Note that (4.21) implies the boundedness of $\|D_j u_m\|_p$, $j = 1, 2, \dots, n$.

4.22. Remark. Corollary 4.13 connects equistrictly convexity and equiuniform convexity in $\mathbb{R}^n$. Such a result is not true in infinite dimensional spaces, since the unit norm ball is never compact in an infinite dimensional normed space, cf. e.g. [Y, p. 85]. For example, consider the strictly convex function $I: L^p(G) \rightarrow \mathbb{R}$,

$$I(u) = \int_G F(x, u(x)) \, dx,$$

where F is as in 4.17. By Theorem 4.18, the F -norm $I^{1/p}$ is not uniformly convex though the singleton $\{I\}$ is equistrictly convex.

4.23. *An application to the Calculus of Variations.* For a bounded open set G in $\mathbb{R}^n$ fix $u_0 \in W_{p,0}^1(G)$. Write

$$F = \{u: G \rightarrow \mathbb{R} \mid u - u_0 \in W_{p,0}^1(G)\}.$$

Function v in F is called an *extremal for the variational integral*

$$I(u) = \int_G F(x, \nabla u(x)) \, dx$$

with boundary values u_0 if

$$I(v) = \inf \{I(u) \mid u \in F\}.$$

Suppose that $F: G \times \mathbb{R}^n \rightarrow \mathbb{R}$ is a uniformly convex kernel. Since $\nabla F = \{\nabla u \mid u \in F\}$ is a convex and closed set in $L^p(G)^n$, it contains a unique element ∇v with minimal F -norm. This proves the existence and the uniqueness of the extremal with fixed boundary values u_0 .

5. F-capacity and conformal mappings

For a given norm $f: \mathbb{R}^n \rightarrow \mathbb{R}$ and a constant $p \in (1, \infty)$ we write $F(h) = f(h)^p$, $h \in \mathbb{R}^n$. Suppose that G is a domain in $\mathbb{R}^n$ and that $C_0, C \subset \partial G$ are compact sets. The *F-capacity of C_0 and C relative to G* is the number

$$\text{cap}_F(C_0, C; G) = \inf \left\{ \int_G F(\nabla u) \, dm \mid u \in W(C_0, C; G) \right\},$$

where

$$\begin{aligned} & W(C_0, C; G) \\ &= \{ u \in C(G \cup C_0 \cup C) \mid u \text{ is } \text{ACL}^p \text{ in } G, u|_{C_0} \leq 0 \text{ and } u|_C \geq 1 \}. \end{aligned}$$

We call an F -capacity *conformally invariant* (or *similarity-invariant*) if for each domain $G \subset \mathbb{R}^n$ and all compact sets $C_0, C \subset \partial G$

$$\text{cap}_F(C_0, C; G) = \text{cap}_F(h(C_0), h(C); h(G))$$

for each conformal mapping (similarity) h defined in some open neighbourhood of $\bar{G}$. If f is the euclidean norm in $\mathbb{R}^n$, the F -capacity is the familiar p -capacity, which is known to be conformally invariant in the case $p = n$. This follows also from Theorem 6.3, where a more general situation is studied. In this section we shall prove the following theorem.

5.1. Theorem. *The F -capacity is similarity-invariant if and only if there exists $\lambda > 0$ such that $F(h) = \lambda |h|^n$ for all $h \in \mathbb{R}^n$.*

Theorem 5.1 immediately yields

5.2. Corollary. *The F -capacity is conformally invariant if and only if there exists $\lambda > 0$ such that $F(h) = \lambda |h|^n$ for all $h \in \mathbb{R}^n$.*

Since the conformal invariance of F -capacity, $F(h) = \lambda |h|^n$, is obtained from Theorem 6.3, we shall consider only the converse part.

The proof is divided into four lemmas.

Write $g = f|_{S^{n-1}}: S^{n-1} \rightarrow \mathbb{R}$, $\alpha_0 = \min g$ and $\beta_0 = \max g$. Then $0 < \alpha_0 \leq \beta_0 < \infty$. Furthermore, $\alpha_0 = \beta_0$ if and only if for some $\lambda > 0$, $f(h) = \lambda |h|^n$ for all $h \in \mathbb{R}^n$. For $x_0 \in S^{n-1}$ we set

$$L_{x_0} = \{ tx_0 \mid t \in \mathbb{R} \}.$$

Let P_{x_0} be the projection of $\mathbb{R}^n$ onto L_{x_0} , i.e., $P_{x_0}(x) = (x \cdot x_0)x_0$.

5.3. Lemma. Suppose that $x_0 \in g^{-1}(\beta_0)$. Then $f(x) \geq f(P_{x_0}(x))$ for all $x \in \mathbb{R}^n$.

Proof. Write $y_0 = x_0/\beta_0$; then $f(y_0) = 1$. Fix $x \in \mathbb{R}^n$. We may assume that $f(x) = 1$ and that $P_{x_0}(x) = \mu y_0$, $\mu > 0$. Suppose that $\mu > 1$. Since

$$\frac{1}{\beta_0} < \frac{\mu}{\beta_0} = |\mu y_0| = |P_{x_0}(x)| = |x \cdot x_0| |x_0| \leq |x|,$$

it follows by elementary geometry that there are $y \in S^{n-1}(\beta_0^{-1})$ and $t \in (0,1)$ such that

$$z_0 = ty + (1-t)x \in \{sy_0 \mid s \in (1,\mu)\}.$$

In particular, $f(z_0) > f(y_0) = 1$. On the other hand, $f(y) \leq \beta_0 |y| = 1$; hence the convexity of f yields

$$1 < f(z_0) \leq tf(y) + (1-t)f(x) \leq 1,$$

which is impossible. Thus $\mu \leq 1$ and $f(x) \geq f(P_{x_0}(x))$, as required.

Fix $x_0 \in g^{-1}(\beta_0)$. Let $T: \mathbb{R}^n \rightarrow \mathbb{R}^n$ be an orthogonal mapping such that $Te_1 = x_0$. Write $G = TQ(1)$, $C_0 = T\{x \in \partial Q(1) \mid x_1 = -1\}$ and $C = T\{x \in \partial Q(1) \mid x_1 = 1\}$.

5.4. Lemma. $\text{cap}_F(C_0, C; G) = \beta_0^p 2^{n-p}$.

Proof. If $u \in W(C_0, C; G)$, then Lemma 5.3 yields

$$(5.5) \quad F(\nabla u) \geq F(P_{x_0}(\nabla u)) = \beta_0^p |\nabla u \cdot x_0|^p$$

in G . Now Hölder's inequality implies for $x \in TQ_{n-1}(1)$

$$1 \leq \left(\int_{-1}^1 |\nabla u(x + tx_0) \cdot x_0| dt \right)^p \leq 2^{p-1} \int_{-1}^1 |\nabla u(x + tx_0) \cdot x_0|^p dt,$$

and hence by (5.5) and by Fubini's theorem

$$\begin{aligned}
\int_G F(\nabla u) \, dm &\geq \beta_0^p \int_{TQ_{n-1}(1)} \int_{-1}^1 |\nabla u(x + tx_0) \cdot x_0|^p \, dt \, dm_{n-1} \\
&\geq \beta_0^p m_{n-1}(Q_{n-1}(1)) 2^{1-p} = \beta_0^p 2^{n-p} .
\end{aligned}$$

Next set $v(x) = \frac{1}{2}(x_1 + 1)$. Then $u_0 = v \circ T^{-1}$ belongs to $W(C_0, C; G)$ and

$$\nabla u_0(x) = T \nabla v(T^{-1}x) = \frac{1}{2}x_0 \in L_{x_0}$$

in G . This yields

$$\int_G F(\nabla u_0) \, dm = \beta_0^p \int_G 2^{-p} \, dm = \beta_0^p 2^{n-p}$$

and thus

$$\text{cap}_F(C_0, C; G) = \beta_0^p 2^{n-p} ,$$

as required.

5.6. Lemma. *If F-capacity is similarity-invariant, then $p = n$.*

Proof. Let $h(x) = \lambda x$, $\lambda \neq 0$, be a dilation. Set $G' = h(G)$, $C'_0 = h(C_0)$ and $C' = h(C)$. Now $u \in W(C_0, C; G)$ if and only if $u \circ h^{-1} \in W(C'_0, C'; G')$. Fix $u \in W(C_0, C; G)$. Then

$$\nabla(u \circ h^{-1})(x) = \frac{1}{\lambda} \nabla u(h^{-1}(x))$$

in G' and thus

$$\begin{aligned}
\int_{G'} F(\nabla(u \circ h^{-1})) \, dm &= |\lambda|^{-p} \int_G |\lambda|^n F(\nabla u) \, dm \\
&= |\lambda|^{n-p} \int_G F(\nabla u) \, dm .
\end{aligned}$$

So Lemma 5.4 yields for each $\lambda \neq 0$

$$\begin{aligned}\beta_0^p 2^{n-p} &= \text{cap}_F(C'_0, C'; G') = |\lambda|^{n-p} \text{cap}_F(C_0, C; G) \\ &= |\lambda|^{n-p} \beta_0^p 2^{n-p} ;\end{aligned}$$

hence $p = n$ as desired.

The next lemma completes the proof of Theorem 5.1.

5.7. Lemma. If $\alpha_0 < \beta_0$, then F-capacity is not similarity-invariant.

Proof. Fix $y_0 \in g^{-1}(\alpha_0)$. Let $O: \mathbb{R}^n \rightarrow \mathbb{R}^n$ be an orthogonal mapping with $O(x_0) = y_0$. Write $G' = O(G)$, $C'_0 = O(C_0)$ and $C' = O(C)$. Suppose that u_0 is as in the proof of Lemma 5.4. Set $v_0 = u_0 \circ O^{-1}$. Then v_0 belongs to $W(C'_0, C'; G')$ and

$$\nabla v_0(x) = O \nabla u_0(O^{-1}(x)) = \frac{1}{2} y_0$$

in G' ; hence

$$\begin{aligned}\text{cap}_F(C'_0, C'; G') &\leq \int_{G'} F(\nabla v_0) \, dm = \alpha_0^p \int_{G'} \left| \frac{1}{2} y_0 \right|^p \, dm \\ &= \alpha_0^p 2^{n-p} < \beta_0^p 2^{n-p} = \text{cap}_F(C_0, C; G) ,\end{aligned}$$

which contradicts the similarity invariance.

6. Conformally invariant variational integrals

Suppose that G is an open set in $\mathbb{R}^n$ and that $F: G \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$ is a C-function in $G \times (\mathbb{R} \times \mathbb{R}^n)$. Suppose, furthermore, that there exist constants $\alpha \in [0, \infty)$, $\beta \in (0, \infty)$ and $p \in [1, \infty)$ such that for a.e. $x \in G$

$$(6.1) \quad 0 \leq F(x, u, h) \leq \alpha |u|^p + \beta |h|^p$$

for all $(u, h) \in \mathbb{R} \times \mathbb{R}^n$. Setting $F(x, u, h) = \beta |h|^p$ for $x \notin G$ we may assume that $F: \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$.

If $D \subset \mathbb{R}^n$ is an open set and if $u \in W_p^1(D)$, then the variational integral

$$I_F(u) = \int_D F(x, u(x), \nabla u(x)) \, dx$$

exists and is finite. The variational integral I_F is *conformally invariant* (or *similarity-invariant*) if for each open set $D \subset \mathbb{R}^n$ and for every conformal (similarity) $f: D \rightarrow \tilde{D} = f(D)$

$$I_F(u) = I_F(u \circ f)$$

i.e.,

$$\int_{\tilde{D}} F(y, u(y), \nabla u(y)) \, dy = \int_D F(f(x), u \circ f(x), \nabla(u \circ f)(x)) \, dx$$

whenever $u \in W_p^1(\tilde{D})$.

6.2. Remark. The growth restriction (6.1) is appropriate to guarantee $I_F(u)$ finite whenever $u \in W_p^1(D)$, see [Kr, p. 27].

The following theorem gives the structure of conformally invariant or similarity-invariant variational kernels and generalizes a result of M. Grüter's [Gr].

6.3. Theorem. Suppose that a C-function $F: \mathbb{R}^n \times (\mathbb{R} \times \mathbb{R}^n) \rightarrow \mathbb{R}$ satisfies (6.1). Then I_F is conformally invariant (or similarity-invariant) if and only if there is a C-function $k: \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}$ such that for a.e. $x \in \mathbb{R}^n$

$$F(x, u, h) = k(x, u) |h|^n$$

for each $u \in \mathbb{R}$ and $h \in \mathbb{R}^n$.

Proof. To show the sufficiency, fix $D \subset \mathbb{R}^n$ and a conformal

mapping $f: D \rightarrow f(D)$. Pick $u \in W_p^1(f(D))$. Using the change of variable, see [GLM, 6.8; V, p. 113; G, Theorem 6], and the conformality of f , we obtain

$$\begin{aligned}
 \int_{f(D)} F(y, u(y), \nabla u(y)) \, dy &= \int_D |J(x, f)| \, k(f(x), u \circ f(x)) \, |\nabla u(f(x))|^n \, dx \\
 &= \int_D k(f(x), u(f(x))) \, |f'(x)|^n \, |\nabla u(f(x))|^n \, dx \\
 &= \int_D k(f(x), u(f(x))) \, |f'(x)^* \nabla u(f(x))|^n \, dx \\
 &= \int_D k(f(x), u(f(x))) \, |\nabla(u \circ f)(x)|^n \, dx \\
 &= \int_D F(f(x), u \circ f(x), \nabla(u \circ f)(x)) \, dx,
 \end{aligned}$$

as desired.

To show the converse, we first prove some auxiliary results.

Suppose that G is an open set in $\mathbb{R}^n$ and that $f: G \rightarrow f(G) = G'$ is conformal. Suppose, furthermore, that

$$I_F(u) = I_F(u \circ f)$$

for each $u \in W_p^1(f(D))$ whenever D is a bounded open set in G .

6.4. Lemma. For a.e. $x \in G$

$$F(f(x), u, f'(x)^* h) = |J(x, f)| \, F(f(x), u, h)$$

for each $(u, h) \in \mathbb{R} \times \mathbb{R}^n$.

Proof. Since the problem is local, we may assume that f is defined in an open set containing $\bar{G}$ and that G is bounded. If $D \subset G$ is open, we obtain by a change of variable (see [GLM, 6.8; V, p. 113; G, Theorem 6])

$$\begin{aligned}
\int_D F(f(x), u \circ f(x), \nabla(u \circ f)(x)) \, dx &= \int_{f(D)} F(y, u(y), \nabla u(y)) \, dy \\
&= \int_D |J(x, f)| F(f(x), u \circ f(x), \nabla u(f(x))) \, dx
\end{aligned}$$

for each $u \in W_p^1(f(D))$. Fix $c \in \mathbb{R}$ and $h \in \mathbb{R}^n$. Write $u(x) = h \cdot x + c$. Then $u \in W_p^1(f(D))$ for each open $D \subset G$ and $\nabla u(x) = h$. Choose M, N and $K \in \mathbb{R}$ such that

$$|f'(x)|^n < M, \quad \alpha |u(f(x))|^p + \beta |h|^p < K$$

and

$$\alpha |u(f(x))|^p + \beta |f'(x) * h|^p < N$$

for every $x \in G$. We show that for a.e. $x \in G$

$$(6.5) \quad F(f(x), u(f(x)), f'(x) * h) = |J(x, f)| F(f(x), u(f(x)), h).$$

Fix $\delta > 0$ and write

$$\begin{aligned}
U_\delta &= \{x \in G \mid F(f(x), u(f(x)), f'(x) * h) \\
&\quad > |J(x, f)| F(f(x), u(f(x)), h) + \delta\}.
\end{aligned}$$

If $m(U_\delta) > 0$, pick an open set $U \subset G$ such that $U_\delta \subset U$ and $m(U \setminus U_\delta) < (\delta m(U_\delta))/MK$. Then

$$\begin{aligned}
&\int_{U \setminus U_\delta} |J(x, f)| F(f(x), u(f(x)), h) \, dx \\
&= \int_{U \setminus U_\delta} |f'(x)|^n F(f(x), u(f(x)), h) \, dx \\
&\leq MK m(U \setminus U_\delta) < \delta m(U_\delta).
\end{aligned}$$

Hence

$$\begin{aligned}
& \int_U F(f(x), u(f(x)), \nabla(u \circ f)(x)) \, dx = \int_U F(f(x), u(f(x)), f'(x)^* h) \, dx \\
& \geq \int_{U_\delta} F(f(x), u(f(x)), f'(x)^* h) \, dx \\
& > \delta m(U_\delta) + \int_{U_\delta} |J(x, f)| F(f(x), u(f(x)), h) \, dx \\
& > \int_U |J(x, f)| F(f(x), u(f(x)), h) \, dx = \int_{f(U)} F(y, u(y), \nabla u(y)) \, dy,
\end{aligned}$$

which contradicts the invariance of I_F . Hence $m(U_\delta) = 0$. Similarly, write

$$\begin{aligned}
V_\delta &= \{x \in G \mid F(f(x), u(f(x)), f'(x)^* h) + \delta \\
&\quad < |J(x, f)| F(f(x), u(f(x)), h)\}.
\end{aligned}$$

If $m(V_\delta) > 0$, pick an open set $V \subset G$ such that $V_\delta \subset V$ and $m(V \setminus V_\delta) < (\delta m(V_\delta))/N$. From the estimate

$$\begin{aligned}
& \int_{V \setminus V_\delta} F(f(x), u(f(x)), f'(x)^* h) \, dx \\
& \leq \int_{V \setminus V_\delta} (\alpha |u(f(x))|^P + \beta |f'(x)^* h|^P) \, dx \leq Nm(V \setminus V_\delta) < \delta m(V_\delta)
\end{aligned}$$

we obtain $m(V_\delta) = 0$ as above. Since $m(U_\delta) = 0 = m(V_\delta)$ for each $\delta > 0$, the equation (6.5) follows.

To complete the proof, write

$$G' = \{x \in G \mid (u, h) \mapsto F(f(x), u, h) \text{ is continuous}\}.$$

Then $m(G \setminus G') = 0$, and if we set

$$\begin{aligned}
A_{c, h} &= \{x \in G' \mid F(f(x), f(x) \cdot h + c, f'(x)^* h) \\
&= |J(x, f)| F(f(x), f(x) \cdot h + c, h)\},
\end{aligned}$$

then $m(G \setminus A_{c,h}) = 0$ by (6.5). Hence $m(G \setminus A) = 0$, where

$$A = \bigcap_{\substack{c \in Q \\ h \in Q^n}} A_{c,h}.$$

Since

$$\begin{aligned} A &= \{x \in G' \mid F(f(x), f(x) \cdot h + c, f'(x)^*h) \\ &= |J(x, f)| F(f(x), f(x) \cdot h + c, h) \\ &\text{for all } c \in Q \text{ and } h \in Q^n\}, \end{aligned}$$

the continuity of $(v, h) \mapsto F(f(x), v, h)$ for $x \in A$ and the density of $Q \times Q^n$ in $\mathbb{R} \times \mathbb{R}^n$ imply

$$\begin{aligned} A &= \{x \in G' \mid F(f(x), f(x) \cdot h + c, f'(x)^*h) \\ &= |J(x, f)| F(f(x), f(x) \cdot h + c, h) \\ &\text{for all } c \in \mathbb{R} \text{ and } h \in \mathbb{R}^n\}. \end{aligned}$$

Since the function $c \mapsto f(x) \cdot h + c$ is surjective onto $\mathbb{R}$ for fixed $x \in A$ and $h \in \mathbb{R}^n$, we obtain

$$\begin{aligned} A &= \{x \in G' \mid F(f(x), u, f'(x)^*h) = |J(x, f)| F(f(x), u, h) \\ &\text{for all } u \in \mathbb{R} \text{ and } h \in \mathbb{R}^n\}, \end{aligned}$$

and the proof is complete.

6.6. Corollary. *If I_F is similarity-invariant, then for a.e. $x \in \mathbb{R}^n$*

$$F(x, u, \lambda h) = |\lambda|^n F(x, u, h)$$

for every $\lambda \in \mathbb{R}$ and $(u, h) \in \mathbb{R} \times \mathbb{R}^n$.

Proof. Choose $f(x) = \lambda x$ in Lemma 6.4. Then for a.e. $x \in \mathbb{R}^n$

$$F(x, u, \lambda h) = |\lambda|^n F(x, u, h)$$

for each $(u, h) \in \mathbb{R} \times \mathbb{R}^n$. As in the proof for Lemma 6.4, the G-function property yields the desired result.

Proof for the necessity in 6.3. Fix $h_0 \in S^{n-1}$. For $x \in \mathbb{R}^n$ and $u \in \mathbb{R}$ set

$$k(x, u) = F(x, u, h_0) .$$

Pick $h \in S^{n-1}$ and choose an orthogonal mapping $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that $f(h_0) = h$. Then

$$f'(x) \circ h = f^{-1}(h) = h_0$$

for each $x \in \mathbb{R}^n$. Hence Lemma 6.4 implies that for a.e. $x \in \mathbb{R}^n$

$$F(f(x), u, h_0) = F(f(x), u, h)$$

for all $u \in \mathbb{R}$. Observe that $|J(x, f)| = 1$. Now for a.e. $x \in \mathbb{R}^n$

$$k(x, u) = F(x, u, h_0) = F(x, u, h)$$

for all $u \in \mathbb{R}$. Write

$$G' = \{ x \in \mathbb{R}^n \mid (u, h) \mapsto F(x, u, h) \text{ is continuous and}$$

$$F(x, u, \lambda h) = |\lambda|^n F(x, u, h) \text{ for all}$$

$$\lambda, u \in \mathbb{R} \text{ and } h \in \mathbb{R}^n \}$$

and

$$A_h = \{ x \in G' \mid F(x, u, h) = k(x, u) \text{ for each } u \in \mathbb{R} \} ,$$

Then $m(\mathbb{R}^n \setminus A_h) = 0$ for every $h \in S^{n-1}$, since $m(\mathbb{R}^n \setminus G') = 0$ by Lemma 6.4. Using similar reasoning to the proof of Lemma 6.4 we obtain $m(\mathbb{R}^n \setminus A) = 0$, where

$$A = \{x \in G' \mid F(x, u, h) = k(x, u) \text{ for all } u \in \mathbb{R} \text{ and } h \in S^{n-1}\}.$$

Hence the homogeneity property of F in G' implies that

$$A = \{x \in G' \mid F(x, u, h) = k(x, u) |h|^n \text{ for all } u \in \mathbb{R} \text{ and } h \in \mathbb{R}^n\},$$

which completes the proof.

6.7. Corollary. Suppose that a C-function $F: \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$ satisfies (6.1) for $p = n$. Then I_F is similarity-invariant if and only if there is a C-function $k: \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}$ such that for a.e. $x \in \mathbb{R}^n$

$$F(x, u, h) = k(x, u) |h|^n$$

for each $u \in \mathbb{R}$ and $h \in \mathbb{R}^n$ and that for each $u \in \mathbb{R}$ the function $x \mapsto k(x, u)$ belongs to $L^\infty(\mathbb{R}^n)$.

Proof. The growth restriction (6.1) implies that for a.e. $x \in \mathbb{R}^n$

$$0 \leq k(x, u) \leq \frac{\alpha |u|^n}{|h|^n} + \beta$$

for every $u \in \mathbb{R}^n$ and $h \in \mathbb{R}^n$, $h \neq 0$. Letting $|h| \rightarrow \infty$ we obtain the desired result.

6.8. Remark. The case $p \neq n$ in (6.1) is not interesting. In fact, if I_F is similarity-invariant and $p < n$, it is easily seen that $F = 0$. The same result is obtained if $p > n$ and $\alpha = 0$.

If for a.e. $x \in \mathbb{R}^n$, $F(x, u, h) = F(x, v, h)$ for every $u, v \in \mathbb{R}$ and $h \in \mathbb{R}^n$, we write $F: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ and $F(x, h) = F(x, u, h)$ for short.

6.9. Corollary. Suppose that a C-function $F: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ satisfies (6.1) for $p = n$. Then I_F is similarity-invariant if and only if there is $k \in L^\infty(\mathbb{R}^n)$ such that for a.e. $x \in \mathbb{R}^n$

$$F(x, h) = k(x) |h|^n$$

for every $h \in \mathbb{R}^n$.

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