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Helsinki, January 2015

Sauli Lindberg

Contents

1. Introduction

Consider a mapping $u = u_1 + iu_2 \in W^{1,2}_{loc}(\mathbb{C}, \mathbb{C})$. Its Jacobian is the determinant of the differential matrix,

$$J_u = \det \begin{bmatrix} \partial_x u_1 & \partial_y u_1 \\ \partial_x u_2 & \partial_y u_2 \end{bmatrix} = \partial_x u_1 \partial_y u_2 - \partial_y u_1 \partial_x u_2.$$

This dissertation deals with the Jacobian equation

$$J_u = h,$$

where the data h is given and we look for a solution u. Modern study of the existence and regularity of solutions to the Jacobian equation was initiated by J. Moser in [Mos65]. Later contributions can be found, for instance, in [Rei72], [Dac81], [DM90], [Ye94], [RY96], [BK02], [McM98], [CDK09] and the references contained therein.

In incompressible nonlinear elasticity the equation $J_u = 1$ is used to express the incompressibility of a material (see e.g. [Bal77], [Cia98], [Le Dre85] and [LO81]). The Jacobian equation also arises as a special case of the pullback equation (see [CDK12]).

In this dissertation the data h belongs to the Hardy space in the plane, $\mathcal{H}^1(\mathbb{C})$, and we seek a solution u in the homogeneous Sobolev space

$$\dot{W}^{1,2}(\mathbb{C}, \mathbb{C}) := \{u \in L^1_{loc}(\mathbb{C}, \mathbb{C}) \colon \mathrm{D}u \in L^2(\mathbb{C}, \mathbb{R}^{2\times 2})\}.$$

R. Coifman, P.-L. Lions, Y. Meyer and S. Semmes proved in the seminal paper [CLMS93] the following result (in $\mathbb{R}^n$ for a general $n \geq 2$):

Theorem 1.1. *Let $u = u_1 + iu_2 \in \dot{W}^{1,2}(\mathbb{C}, \mathbb{C})$. Then the Jacobian J_u belongs to $\mathcal{H}^1(\mathbb{C})$ and satisfies the estimate $\|J_u\|_{\mathcal{H}^1} \lesssim \|\nabla u_1\|_{L^2} \|\nabla u_2\|_{L^2}$.*

The authors showed that no proper closed subspace of $\mathcal{H}^1(\mathbb{C})$ contains the Jacobians of all the mappings of $W^{1,2}(\mathbb{C}, \mathbb{C})$. They proceeded to pose the famous problem whether every function in $\mathcal{H}^1(\mathbb{C})$ is the Jacobian of some mapping in $W^{1,2}(\mathbb{C}, \mathbb{C})$. We consider the question in the following form where the inhomogeneous Sobolev space $W^{1,2}(\mathbb{C}, \mathbb{C})$ is replaced by the homogeneous one.

Question 1.2. *Does the Jacobian operator J map $\dot{W}^{1,2}(\mathbb{C}, \mathbb{C})$ onto $\mathcal{H}^1(\mathbb{C})$?*

Question 1.2 is the theme of this dissertation. We are not able to solve the Jacobian equation in full generality, but we reduce solving it to a finite-dimensional problem which is likely to be significantly easier to approach than Question 1.2. Indeed, if Question 1.24 has a positive answer, then the Jacobian operator maps $\dot{W}^{1,2}(\mathbb{C}, \mathbb{C})$ onto $\mathcal{H}^1(\mathbb{C})$! In the main result of the dissertation, Theorem 1.27, we present results that make a positive answer to Question 1.24 appear plausible. In Theorem 1.26 we tie together some of the key elements of the dissertation and put Theorem 1.27 into context.

Let us recall some of the notions relevant to the study of the Jacobian equation. When $u \in \dot{W}^{1,2}(\mathbb{C}, \mathbb{C})$, the Wirtinger derivatives of u are defined by

$$u_z := \frac{1}{2}(\partial_x - i\partial_y)(u_1 + iu_2) \quad \text{and} \quad u_{\bar{z}} := \frac{1}{2}(\partial_x + i\partial_y)(u_1 + iu_2).$$

The Beurling transform $\mathcal{S}\colon L^2(\mathbb{C}, \mathbb{C}) \to L^2(\mathbb{C}, \mathbb{C})$ (see §2.6) is used to write $u_z = \mathcal{S}u_{\bar{z}}$, and since $\mathcal{S}$ is an isometry in $L^2(\mathbb{C}, \mathbb{C})$, it follows that

$$\int_{\mathbb{C}} |u_z|^2 = \int_{\mathbb{C}} |\mathcal{S}u_{\bar{z}}|^2 = \int_{\mathbb{C}} |u_{\bar{z}}|^2.$$

By combining this with the identity $|\mathrm{D}u|^2 = 2(|u_z|^2 + |u_{\bar{z}}|^2)$ (where $|\mathrm{D}u|$ is the Hilbert-Schmidt norm) we obtain $\|\mathrm{D}u\|_{L^2} = 2\|u_{\bar{z}}\|_{L^2}$.

We endow $\dot{W}^{1,2}(\mathbb{C}, \mathbb{C})$ with the seminorm given by

$$\|u\|_{\dot{W}^{1,2}} := \|u_{\bar{z}}\|_{L^2}$$

in order to make the linear operator $u \mapsto u_{\bar{z}}\colon \dot{W}^{1,2}(\mathbb{C}, \mathbb{C}) \to L^2(\mathbb{C}, \mathbb{C})$ an isometry. Denoting $f := u_{\bar{z}} \in L^2(\mathbb{C}, \mathbb{C})$ we use the Beurling transform to write

$$J_u = |u_z|^2 - |u_{\bar{z}}|^2 = |\mathcal{S}f|^2 - |f|^2.$$

This allows us to use the special properties of the Beurling transform in the study of Jacobians and effortlessly transfer arguments from $L^2(\mathbb{C}, \mathbb{C})$ to $\dot{W}^{1,2}(\mathbb{C}, \mathbb{C})$ and reversely.

In §3 the Jacobian equation is studied in the model case of compactly supported, radially symmetric data. A measurable function $h\colon \mathbb{C} \to \mathbb{R}$ is said to be *radial* if it is of the form $h(z) = h(|z|)$. Likewise, a measurable mapping $u\colon \mathbb{C} \to \mathbb{C}$ is called a *radial stretching* if it is of the form

$$u(z) := \rho(|z|)\frac{z}{|z|},$$

where $\rho(r) \in \mathbb{R}$ for every $r \in [0, \infty)$. It is natural to ask whether for a radial data h there always exists a solution u that is a radial stretching. As one of the two main theorems of §3 we prove the following result.

Theorem 1.3. *Suppose $h \in \mathcal{H}^1(\mathbb{C})$ is compactly supported, Lipschitz continuous and radial. Then the following conditions are equivalent.*

(*i*) *There exists a radial stretching $u \in \dot{W}^{1,2}(\mathbb{C}, \mathbb{C})$ that satisfies the Jacobian equation $J_u = h$.*

(*ii*) $\int_{B(0,r)} h \geq 0$ *for every $r > 0$.*

In Theorem 3.5 we show that Theorem 1.3 is optimal in the scale of C^α spaces: Lipschitz continuity cannot be replaced by C^α continuity for any $0 < \alpha < 1$.

In order to allow the integrals $\int_{B(0,r)} J_u$ to change sign we generalize radial stretchings by assuming that u is of the form

$$u(z) := \rho(|z|)\frac{\gamma(z)}{|\gamma(z)|}, \tag{1.1}$$

where $\gamma(z) \neq 0$ a.e. Note that radial stretchings are a special case where $\gamma(z) = z$. In §3.2 the following analogue of Theorem 1.3 is proved.

Theorem 1.4. *Suppose $h \in \mathcal{H}^1(\mathbb{C})$ is compactly supported, Lipschitz continuous and radial. Then the following conditions are equivalent.*
(i) There exists a mapping $u \in \dot{W}^{1,2}(\mathbb{C}, \mathbb{C})$ of the form (1.1) that satisfies the Jacobian equation $J_u = h$.
(ii) If $r > 0$ and $\int_{B(0,r)} h(z)\, dz = 0$, then $h(r) = 0$.

Theorems 1.3 and 1.4 give limitations to solutions of the Jacobian equation with radial symmetry properties. If it turns out that the Jacobian operator does not map $\dot{W}^{1,2}(\mathbb{C}, \mathbb{C})$ onto $\mathcal{H}^1(\mathbb{C})$, it is hoped that Theorems 1.3 and 1.4 are of help in producing a counterexample. However, the majority of the results in this dissertation are formulated in an effort to provide a positive answer to Question 1.2. We next start describing the general strategy used in this dissertation.

We briefly discuss the work of G. Cupini, B. Dacorogna and O. Kneuss on the Jacobian equation and its relation to Question 1.2. Whereas in the earlier papers on the Jacobian equation mentioned above the data was assumed to be positive, the crucial novelty in [CDK09] is that the data is allowed to change sign. This makes the study of the Jacobian equation much more difficult for various reasons; it prevents, for instance, the flow method of Moser from being used. The relaxation on the sign of the data is, however, vital in our setting since the assumption that the data h belongs to $\mathcal{H}^1(\mathbb{C})$ implies that $\int_{\mathbb{C}} h = 0$.

In [Kne12] O. Kneuss used ideas related to those in [CDK09] to study the case of smooth, compactly supported data whose integral over $\mathbb{R}^n$ vanishes. The following important result follows directly from [Kne12, Theorem 1] (see §4.1).

Theorem 1.5. *The range of the operator $J \colon \dot{W}^{1,2}(\mathbb{C}, \mathbb{C}) \to \mathcal{H}^1(\mathbb{C})$ is a dense subset of $\mathcal{H}^1(\mathbb{C})$.*

Theorem 1.5 and a weak continuity argument reduce Question 1.2, in the positive direction, to proving a suitable *a priori* estimate (1.2) for energy-minimal solutions that we next define.

Definition 1.6. If a mapping $u \in \dot{W}^{1,2}(\mathbb{C}, \mathbb{C})$ satisfies

$$\int_{\mathbb{C}} |u_{\bar{z}}|^2 = \min \left\{ \int_{\mathbb{C}} |v_{\bar{z}}|^2 \colon v \in \dot{W}^{1,2}(\mathbb{C}, \mathbb{C}),\ J_v = J_u \right\},$$

then we call u an *energy minimizer* or an *energy-minimal solution.*

If the Jacobian equation $J_u = h$ has a solution, then it has an energy-minimal solution; for a proof of this basic result see Proposition 4.2.

Remark 1.7. Suppose every energy minimizer $u \in \dot{W}^{1,2}(\mathbb{C}, \mathbb{C})$ satisfies

$$\int_{\mathbb{C}} |u_{\bar{z}}|^2 \lesssim \|J_u\|_{\mathcal{H}^1}. \tag{1.2}$$

Then J maps $\dot{W}^{1,2}(\mathbb{C}, \mathbb{C})$ onto $\mathcal{H}^1(\mathbb{C})$. We present a proof of this standard fact in §4.2.

It appears very difficult to control the norms of energy minimizers directly. We therefore use a rather indirect way to study energy minimizers. The minimization problem we are concerned with is the following one:

$$\text{minimize} \int_{\mathbb{C}} |u_{\bar{z}}|^2 \text{ subject to the constraint } J_u = h, \tag{1.3}$$

where $h \in \mathcal{H}^1(\mathbb{C})$ is given. We approach the problem via Lagrange multipliers.

The Lagrange multipliers related to (1.3) belong to $\mathrm{BMO}(\mathbb{C})$. By C. Fefferman's famous theorem, $\mathrm{BMO}(\mathbb{C})$ is, suitably interpreted, the dual of $\mathcal{H}^1(\mathbb{C})$ (see §2.5). Likewise, by a result of R.R. Coifman and G. Weiss, $\mathcal{H}^1(\mathbb{C})$ is the dual of $\mathrm{VMO}(\mathbb{C})$, that is, of the closure of $C_0^\infty(\mathbb{C}, \mathbb{C})$ in $\mathrm{BMO}(\mathbb{C})$. When $b \in \mathrm{BMO}(\mathbb{C})$ and $h \in \mathcal{H}^1(\mathbb{C})$, we denote the dual pairing of b and h by

$$\int_{\mathbb{C}}^{*} bh := \langle b, h \rangle_{\mathrm{BMO}-\mathcal{H}^1}.$$

Definition 1.8. A function $b \in \mathrm{BMO}(\mathbb{C})$ is called a *Lagrange multiplier* for $u \in \dot{W}^{1,2}(\mathbb{C}, \mathbb{C})$ if

$$\frac{d}{d\epsilon} \int_{\mathbb{C}}^{*} bJ_{u+\epsilon\phi} \bigg|_{\epsilon=0} = \frac{d}{d\epsilon} \int_{\mathbb{C}} |(u + \epsilon\phi)_{\bar{z}}|^2 \bigg|_{\epsilon=0}$$

for every $\phi \in \dot{W}^{1,2}(\mathbb{C}, \mathbb{C})$, that is,

$$\int_{\mathbb{C}}^{*} b\, \mathbf{Re}(u_z \overline{\phi_z} - u_{\bar{z}} \overline{\phi_{\bar{z}}}) = \mathbf{Re} \int_{\mathbb{C}} u_{\bar{z}} \overline{\phi_{\bar{z}}} \quad \text{for all } \phi \in \dot{W}^{1,2}(\mathbb{C}, \mathbb{C}). \tag{1.4}$$

The definition of Lagrange multipliers is discussed more thoroughly in §4.2-4.3. The basic idea behind using Lagrange multipliers is to introduce functional analytic methods in the study of the problem. Furthermore, if the Lagrange multipliers are uniformly bounded in the norm $\|\cdot\|_{\mathrm{BMO}}$, then they can be used to find mappings that satisfy (1.2). Indeed, suppose $b \in \mathrm{BMO}(\mathbb{C})$ is a Lagrange multiplier for $u \in \dot{W}^{1,2}(\mathbb{C}, \mathbb{C})$. Setting $\phi = u$ in condition (1.4) we get

$$\int_{\mathbb{C}} |u_{\bar{z}}|^2 = \int_{\mathbb{C}}^{*} bJ_u \lesssim \|b\|_{\mathrm{BMO}} \|J_u\|_{\mathcal{H}^1}. \tag{1.5}$$

The following proposition summarizes the foregoing discussion.

Proposition 1.9. *Suppose every energy minimizer $u \in \dot{W}^{1,2}(\mathbb{C}, \mathbb{C})$ has a Lagrange multiplier $b \in \mathrm{BMO}(\mathbb{C})$ and the norms of the Lagrange multipliers are uniformly bounded. Then every energy minimizer satisfies*

$$\int_{\mathbb{C}} |u_{\bar{z}}|^2 \lesssim \|J_u\|_{\mathcal{H}^1}$$

and the Jacobian operator maps $\dot{W}^{1,2}(\mathbb{C}, \mathbb{C})$ onto $\mathcal{H}^1(\mathbb{C})$.

In §4.3 we indicate why producing Lagrange multipliers is a non-trivial task in our setting. We next discuss the method of constructing uniformly norm-bounded Lagrange multipliers in this dissertation. We prefer to treat Lagrange multipliers mostly in the setting of $L^2(\mathbb{C}, \mathbb{C})$ and the nonlinear operator $f \mapsto |\mathcal{S}f|^2 - |f|^2$ instead of $\dot{W}^{1,2}(\mathbb{C}, \mathbb{C})$ and the Jacobian. We thus write condition (1.4) in terms of L^2 functions.

Definition 1.10. A function $b \in \mathrm{BMO}(\mathbb{C})$ is called a *Lagrange multiplier* for $f \in L^2(\mathbb{C}, \mathbb{C})$ if

$$\int_{\mathbb{C}}^{*} b\,\mathbf{Re}(\mathcal{S}f\overline{\mathcal{S}\varphi} - f\overline{\varphi}) = \mathbf{Re}\int_{\mathbb{C}} f\overline{\varphi} \quad \text{for all } \varphi \in L^2(\mathbb{C}, \mathbb{C}).$$

In further analogy, we call $f \in L^2(\mathbb{C}, \mathbb{C})$ an *energy minimizer* or an *energy-minimal solution* if

$$\int_{\mathbb{C}} |f|^2 = \min\left\{\int_{\mathbb{C}} |g|^2 \colon |\mathcal{S}g|^2 - |g|^2 = |\mathcal{S}f|^2 - |f|^2\right\}.$$

We will construct Lagrange multipliers by using the norm that we next define.

Definition 1.11. When $b \in \mathrm{BMO}(\mathbb{C})$, we define

$$\|b\|_{\mathrm{BMO}_{\mathcal{S}}} := \sup_{f \in L^2(\mathbb{C},\mathbb{C})\setminus\{0\}} \frac{\int_{\mathbb{C}}^{*} b(|\mathcal{S}f|^2 - |f|^2)}{\int_{\mathbb{C}} |f|^2} = \sup_{\|f\|_{L^2}=1} \int_{\mathbb{C}}^{*} b(|\mathcal{S}f|^2 - |f|^2). \tag{1.6}$$

The corresponding dual norm of $\mathcal{H}^1(\mathbb{C})$ is denoted by

$$\|h\|_{\mathcal{H}^1_{\mathcal{S}}} := \sup\left\{\int_{\mathbb{C}}^{*} bh \colon b \in \mathrm{VMO}(\mathbb{C}),\ \|b\|_{\mathrm{BMO}_{\mathcal{S}}} \le 1\right\}.$$

The norm $\|\cdot\|_{\mathrm{BMO}_{\mathcal{S}}}$ was defined, equivalently modulo a multiplicative constant and in a different appearance, in [CLMS93]. The authors proved $\|\cdot\|_{\mathrm{BMO}_{\mathcal{S}}}$ to be equivalent to the standard norm $\|\cdot\|_{\mathrm{BMO}}$. In §2.8 we give a different proof of the equivalence of the two norms by showing that $\|b\|_{\mathrm{BMO}_{\mathcal{S}}} = \|\mathcal{S}b - b\mathcal{S}\|_{L^2 \to L^2}$ for every $b \in \mathrm{BMO}(\mathbb{C})$. Unlike the proof given in [CLMS93], our proof does not, however, appear to generalize readily to higher dimensions.

Definition 1.12. When $\mathrm{VMO}(\mathbb{C})$ and $\mathrm{BMO}(\mathbb{C})$ are endowed with the norm $\|\cdot\|_{\mathrm{BMO}_{\mathcal{S}}}$, the resulting Banach spaces are denoted by $\mathrm{VMO}_{\mathcal{S}}(\mathbb{C})$ and $\mathrm{BMO}_{\mathcal{S}}(\mathbb{C})$. Likewise, $\mathcal{H}^1(\mathbb{C})$ equipped with the norm $\|\cdot\|_{\mathcal{H}^1_{\mathcal{S}}}$ is denoted by $\mathcal{H}^1_{\mathcal{S}}(\mathbb{C})$.

We denote the closed unit balls of the spaces defined above by $\mathbb{B}_{\mathrm{VMO}_{\mathcal{S}}}$, $\mathbb{B}_{\mathcal{H}^1_{\mathcal{S}}}$ and $\mathbb{B}_{\mathrm{BMO}_{\mathcal{S}}}$ and the unit spheres by $\mathbb{S}_{\mathrm{VMO}_{\mathcal{S}}}$, $\mathbb{S}_{\mathcal{H}^1_{\mathcal{S}}}$ and $\mathbb{S}_{\mathrm{BMO}_{\mathcal{S}}}$.

The following proposition shows that $b \in \mathbb{S}_{\mathrm{BMO}_{\mathcal{S}}}$ is a Lagrange multiplier for $f \in L^2(\mathbb{C}, \mathbb{C}) \setminus \{0\}$ exactly when the leftmost supremum in (1.6) is achieved at f.

Proposition 1.13. *Let $b \in \mathbb{S}_{\mathrm{BMO}_{\mathcal{S}}}$ and $f \in L^2(\mathbb{C}, \mathbb{C})$. The following conditions are equivalent.*

(i) b is a Lagrange multiplier for f.

(ii) $\int_{\mathbb{C}}^{*} b(|\mathcal{S}f|^2 - |f|^2) = \int_{\mathbb{C}} |f|^2$.

If these conditions hold, then f is an energy minimizer.

The assumption $\|b\|_{\mathrm{BMO}_{\mathcal{S}}} = 1$ in Proposition 1.13 and other results assures that the Lagrange multipliers are uniformly bounded in norm. The condition is further discussed in Remark 5.3.

Two natural questions now arise: which functions b are Lagrange multipliers and which mappings f have a (uniformly norm-bounded) Lagrange multiplier? We first treat the former question.

In Theorem 5.10 we prove that $\mathrm{BMO}_{\mathcal{S}}(\mathbb{C})$ is not just isomorphic but *isometrically isomorphic* to the bidual $\mathrm{VMO}_{\mathcal{S}}(\mathbb{C})^{**}$. In particular,

$$\int_{\mathbb{C}}^{*} bh \leq \|b\|_{\mathrm{BMO}_{\mathcal{S}}} \|h\|_{\mathcal{H}^1_{\mathcal{S}}} \quad \text{for all } b \in \mathrm{BMO}_{\mathcal{S}}(\mathbb{C}) \text{ and } h \in \mathcal{H}^1_{\mathcal{S}}(\mathbb{C}). \tag{1.7}$$

Recall that $b \in \mathbb{S}_{\mathrm{BMO}_{\mathcal{S}}}$ is said to be *norm-attaining* if $\int_{\mathbb{C}}^{*} bh = 1$ for some $h \in \mathbb{S}_{\mathcal{H}^1_{\mathcal{S}}}$.

Theorem 1.14. *Let $b \in \mathbb{S}_{\mathrm{BMO}_{\mathcal{S}}}$. The following conditions are equivalent.*
(i) b is a Lagrange multiplier for some $f \in L^2(\mathbb{C}, \mathbb{C}) \setminus \{0\}$.
(ii) b is norm-attaining.
In particular, every $b \in \mathbb{S}_{\mathrm{VMO}_{\mathcal{S}}}$ is a Lagrange multiplier for some $f \in L^2(\mathbb{C}, \mathbb{C}) \setminus \{0\}$.

Remark 1.15. By the Bishop-Phelps theorem, norm-attaining elements are dense in $\mathbb{S}_{\mathrm{BMO}_{\mathcal{S}}}$ (see [Meg98, Theorem 2.11.14]). However, since $\mathrm{VMO}_{\mathcal{S}}(\mathbb{C})$ is not reflexive, $\mathrm{BMO}_{\mathcal{S}}(\mathbb{C})$ is not reflexive either, and so by a fundamental theorem of R.C. James (see [Meg98, Theorem 2.9.4]), some elements of $\mathbb{S}_{\mathrm{BMO}_{\mathcal{S}}}$ are not norm-attaining.

We next characterize mappings possessing a Lagrange multiplier in terms of the norm $\|\cdot\|_{\mathcal{H}^1_{\mathcal{S}}}$. It is easy to show that every $f \in L^2(\mathbb{C}, \mathbb{C})$ satisfies

$$\||\mathcal{S}f|^2 - |f|^2\|_{\mathcal{H}^1_{\mathcal{S}}} \leq \int_{\mathbb{C}} |f|^2 \tag{1.8}$$

(see Lemma 5.1). The following result follows easily from (1.8), (1.7), Proposition 1.13 and the Hahn-Banach theorem (see §5.3).

Theorem 1.16. *Let $f \in L^2(\mathbb{C}, \mathbb{C})$. The following conditions are equivalent.*
(i) f has a Lagrange multiplier $b \in \mathbb{S}_{\mathrm{BMO}_{\mathcal{S}}}$.
(ii) $\int_{\mathbb{C}} |f|^2 = \||\mathcal{S}f|^2 - |f|^2\|_{\mathcal{H}^1_{\mathcal{S}}}$.

Theorems 1.14 and 1.16 make it natural to study mappings in $L^2(\mathbb{C}, \mathbb{C})$ that satisfy the condition $\||\mathcal{S}f|^2 - |f|^2\|_{\mathcal{H}^1_{\mathcal{S}}} = \int_{\mathbb{C}} |f|^2$.

Definition 1.17. We denote

$$L^2_{\mathcal{S}}(\mathbb{C}, \mathbb{C}) = \left\{ f \in L^2(\mathbb{C}, \mathbb{C}) \colon \int_{\mathbb{C}} |f|^2 = \||\mathcal{S}f|^2 - |f|^2\|_{\mathcal{H}^1_{\mathcal{S}}} \right\}$$

and $\mathbb{S}_{L^2_{\mathcal{S}}} := L^2_{\mathcal{S}}(\mathbb{C}, \mathbb{C}) \cap \mathbb{S}_{L^2}$.

Remark 1.18. The class $L^2_{\mathcal{S}}(\mathbb{C}, \mathbb{C})$ is *not* a vector space. We briefly sketch the idea of a proof of this fact. Consider any $f \in L^2_{\mathcal{S}}(\mathbb{C}, \mathbb{C}) \setminus \{0\}$, choose $u \in \dot{W}^{1,2}(\mathbb{C}, \mathbb{C})$ such that $u_{\bar{z}} = f$ (see Lemma 2.26) and set $g := \bar{u}_{\bar{z}}$. Then $f, g \in L^2_{\mathcal{S}}(\mathbb{C}, \mathbb{C})$ but $|\mathcal{S}(f+g)|^2 - |f+g|^2 = J_{u+\bar{u}} = 0$ and $\int_{\mathbb{C}} |f+g|^2 > 0$ so that $f + g \notin L^2_{\mathcal{S}}(\mathbb{C}, \mathbb{C})$.

Theorems 1.14 and 1.16 indicate that the set $\{|\mathcal{S}f|^2 - |f|^2 \colon f \in L^2_{\mathcal{S}}(\mathbb{C}, \mathbb{C})\}$ is rather large, but is it equal to the whole space $\mathcal{H}^1_{\mathcal{S}}(\mathbb{C})$? By homogeneity, the question is equivalent to the question whether $\{|\mathcal{S}f|^2 - |f|^2 \colon f \in \mathbb{S}_{L^2_{\mathcal{S}}}\} = \mathbb{S}_{\mathcal{H}^1_{\mathcal{S}}}$. A partial result is given in the following theorem.

Theorem 1.19. *The following statements hold.*
(i) $\{|\mathcal{S}f|^2 - |f|^2 \colon f \in \mathbb{S}_{L^2_{\mathcal{S}}}\}$ contains all the extreme points of $\mathbb{B}_{\mathcal{H}^1_{\mathcal{S}}}$.
(ii) $\{|\mathcal{S}f|^2 - |f|^2 \colon f \in \mathbb{S}_{L^2_{\mathcal{S}}}\}$ is closed in the relative weak-$*$ topology of $\mathbb{S}_{\mathcal{H}^1_{\mathcal{S}}}$. In particular, $\{|\mathcal{S}f|^2 - |f|^2 \colon f \in \mathbb{S}_{L^2_{\mathcal{S}}}\}$ is norm closed.

Theorem 1.19 gives a fairly strong largeness criterion for $\{|\mathcal{S}f|^2-|f|^2\colon f\in\mathbb{S}_{L^2_\mathcal{S}}\}$ as we shall see below in Remark 1.21 and Theorem 1.27. Since $\mathcal{H}^1_\mathcal{S}(\mathbb{C})$ is a separable dual space (see Proposition 2.15), by a theorem of C. Bessaga and A. Pełczyński every closed, bounded, convex subset of $\mathcal{H}^1_\mathcal{S}(\mathbb{C})$ is the closed convex hull of its extreme points (see Theorem 2.42). This, among other things, motivates study of the duality mapping.

Definition 1.20. The *duality mapping* $D\colon \mathbb{S}_{\mathrm{VMO}_\mathcal{S}}\to\mathbb{S}_{\mathcal{H}^1_\mathcal{S}}$ is defined by

$$D(b):=\left\{h\in\mathbb{S}_{\mathcal{H}^1_\mathcal{S}}\colon \int_\mathbb{C}^* bh=1\right\}.$$

Remark 1.21. When $b\in\mathbb{S}_{\mathrm{VMO}_\mathcal{S}}$, the set $D(b)$ is convex and weak-$*$ compact (in particular, closed and bounded). Furthermore, $h\in D(b)$ is an extreme point of $D(b)$ if and only if h is an extreme point of $\mathbb{S}_{\mathcal{H}^1_\mathcal{S}}$. Thus, by Theorems 1.19 and 2.42, $D(b)$ is the closed convex hull of elements of $\{|\mathcal{S}f|^2-|f|^2\colon f\in\mathbb{S}_{L^2_\mathcal{S}}\}\cap D(b)$.

It turns out that we can say a lot more about $D(b)$ by using the commutator $\mathcal{S}b-b\mathcal{S}\colon L^2(\mathbb{C},\mathbb{C})\to L^2(\mathbb{C},\mathbb{C})$. The basic properties of the operator defined below are treated in §2.8.

Definition 1.22. When $b\in\mathrm{BMO}(\mathbb{C})$, we define the linear operator $K_b\colon L^2(\mathbb{C},\mathbb{C})\to L^2(\mathbb{C},\mathbb{C})$ by

$$K_bf:=\overline{(\mathcal{S}b-b\mathcal{S})\overline{\mathcal{S}f}}.$$

The following proposition, proved in §5.4, relates the operators K_b to Lagrange multipliers; we denote the kernel of $I-K_b$ by $\ker(I-K_b)$.

Proposition 1.23. *Suppose $b\in\mathrm{BMO}(\mathbb{C})$ and $f\in L^2(\mathbb{C},\mathbb{C})$. The following conditions are equivalent.*

(i) $f\in\ker(I-K_b)$.

(ii) b is a Lagrange multiplier for f.

Furthermore, if $b\in\mathbb{S}_{\mathrm{VMO}_\mathcal{S}}$, then $\ker(I-K_b)$ is finite-dimensional and contained in $L^2_\mathcal{S}(\mathbb{C},\mathbb{C})$.

We conclude that when $b\in\mathbb{S}_{\mathrm{VMO}_\mathcal{S}}$, the set

$$\ker(I-K_b)\cap\mathbb{S}_{L^2_\mathcal{S}}=\{f\in\mathbb{S}_{L^2_\mathcal{S}}\colon b\text{ is a Lagrange multiplier for }f\}$$

is the (Euclidean) unit sphere of a finite-dimensional subspace of $L^2(\mathbb{C},\mathbb{C})$. By Propositions 1.13 and 1.23,

$$\{|\mathcal{S}f|^2-|f|^2\colon f\in\ker(I-K_b)\cap\mathbb{S}_{L^2_\mathcal{S}}\}=\{|\mathcal{S}f|^2-|f|^2\colon f\in\mathbb{S}_{L^2_\mathcal{S}}\}\cap D(b).$$

These considerations prompt the following question.

Question 1.24. *Does the operator $f\mapsto|\mathcal{S}f|^2-|f|^2$ map $\ker(I-K_b)\cap\mathbb{S}_{L^2_\mathcal{S}}$ onto $D(b)$ for every $b\in\mathbb{S}_{\mathrm{VMO}_\mathcal{S}}$?*

Remark 1.25. Solving this relatively concrete finite-dimensional problem in the positive would show that the operator $f\mapsto|\mathcal{S}f|^2-|f|^2$ maps $L^2_\mathcal{S}(\mathbb{C},\mathbb{C})$ onto $\mathcal{H}^1_\mathcal{S}(\mathbb{C})$ and thus provide a positive answer to Question 1.2! Indeed, suppose the answer to Question 1.24 is positive. By the Bishop-Phelps theorem, norm-attaining elements are dense in $\mathbb{S}_{\mathcal{H}^1_\mathcal{S}}$, and every norm-attaining element of $\mathbb{S}_{\mathcal{H}^1_\mathcal{S}}$ belongs, by definition, to $D(b)$ for some $b\in\mathbb{S}_{\mathrm{VMO}_\mathcal{S}}$. Now use Theorem 1.19 and homogeneity.

We summarize many of the central points of the dissertation in the following result. Theorem 1.26 is proved by combining Theorem 1.16, the definition of $L^2_{\mathcal{S}}(\mathbb{C},\mathbb{C})$ and Remark 1.25.

Theorem 1.26. *The following statements are equivalent.*

(i) The operator $f \mapsto |\mathcal{S}f|^2 - |f|^2$ *maps* $\ker(I-K_b) \cap \mathbb{S}_{L^2_{\mathcal{S}}}$ *onto* $D(b)$ *for every* $b \in \mathbb{S}_{\mathrm{VMO}_{\mathcal{S}}}$.

(ii) The operator $f \mapsto |\mathcal{S}f|^2 - |f|^2$ *maps* $L^2_{\mathcal{S}}(\mathbb{C},\mathbb{C})$ *onto* $\mathcal{H}^1_{\mathcal{S}}(\mathbb{C})$.

(iii) Every energy minimizer $f \in L^2(\mathbb{C},\mathbb{C})$ *has a Lagrange multiplier* $b \in \mathbb{S}_{\mathrm{BMO}_{\mathcal{S}}}$.

(iv) Every energy minimizer $f \in L^2(\mathbb{C},\mathbb{C})$ *satisfies* $\int_{\mathbb{C}} |f|^2 = \| |\mathcal{S}f|^2 - |f|^2 \|_{\mathcal{H}^1_{\mathcal{S}}}$.

If (i)-(iv) *are true, then the answer to Question 1.2 is positive.*

In §6.3 we prove the following partial result about Question 1.24. The author considers Theorem 1.27 the main theorem of this dissertation. Recall that when $b \in \mathbb{S}_{\mathrm{VMO}_{\mathcal{S}}}$, the convex set $D(b)$ is the closed convex hull of its extreme points.

Theorem 1.27. *Let* $b \in \mathbb{S}_{\mathrm{VMO}_{\mathcal{S}}}$. *The following statements hold.*

(i) $D(b)$ *is contained in a finite-dimensional subspace of* $\mathcal{H}^1_{\mathcal{S}}(\mathbb{C})$.

(ii) $f \mapsto |\mathcal{S}f|^2 - |f|^2 \colon \ker(I-K_b) \cap \mathbb{S}_{L^2_{\mathcal{S}}} \to D(b)$ *is Lipschitz continuous.*

(iii) $\{|\mathcal{S}f|^2 - |f|^2 \colon f \in \ker(I-K_b) \cap \mathbb{S}_{L^2_{\mathcal{S}}}\}$ *contains all the extreme points of* $D(b)$.

(iv) $\{|\mathcal{S}f|^2 - |f|^2 \colon f \in \ker(I-K_b) \cap \mathbb{S}_{L^2_{\mathcal{S}}}\}$ *is closed and path-connected.*

Remark 1.28. *The gist of Theorem 1.27 is path-connectedness.* If $f, g \in \ker(I-K_b) \cap \mathbb{S}_{L^2_{\mathcal{S}}}$ and $|\mathcal{S}f|^2 - |f|^2 \neq |\mathcal{S}g|^2 - |g|^2$, we have

$$\frac{tf + (1-t)g}{\|tf + (1-t)g\|_{L^2}} \in \ker(I-K_b) \cap \mathbb{S}_{L^2_{\mathcal{S}}}$$

for every $t \in [0,1]$. This allows us to form an abundance of paths between points in $\{|\mathcal{S}f|^2 - |f|^2 \colon f \in \ker(I-K_b) \cap \mathbb{S}_{L^2_{\mathcal{S}}}\}$. A similar technique does not work in $\{|\mathcal{S}f|^2 - |f|^2 \colon f \in \mathbb{S}_{L^2_{\mathcal{S}}}\}$ since $L^2_{\mathcal{S}}(\mathbb{C},\mathbb{C})$ is not a vector space (see Remark 1.18).

Question 1.24 is further discussed in §6.3. Question 1.24 and Theorem 1.27 motivate a careful study of the interplay between a mapping $f \in L^2_{\mathcal{S}}(\mathbb{C},\mathbb{C})$ and its Lagrange multiplier $b \in \mathbb{S}_{\mathrm{BMO}_{\mathcal{S}}}$. We present several results on this topic and the local properties of energy minimizers in §7-8.

Local study of energy minimizers leads us to study Jacobians of Sobolev mappings in bounded Lipschitz domains of $\mathbb{C}$. This is the last topic of the dissertation. The following analogue of Question 1.2 was posed by Z.J. Lou, S.Z. Yang and D.J. Song in [LYS05].

Question 1.29. *If* $\Omega \subset \mathbb{C}$ *is a bounded Lipschitz domain and* $h \in \mathcal{H}^1(\mathbb{C})$ *satisfies* $\operatorname{supp}(h) \subset \bar{\Omega}$, *does there exist* $u \in W^{1,2}_0(\Omega,\mathbb{C})$ *with* $J_u = h$ *in* Ω?

We solve this problem in the negative by using a theorem of T. Iwaniec and V. Šverák on mappings with integrable distortion (see §2.2). The solution is presented in §8.3.

2. Preliminaries

In this chapter we set definitions and present, mostly without proofs, some of the results relevant to the study of the Jacobian equation. Sections 2.1-2.2 deal with Sobolev spaces in the plane, whereas §2.3-2.5 treat VMO($\mathbb{C}$), $\mathcal{H}^1(\mathbb{C})$, BMO($\mathbb{C}$) and the interplay between them. In §2.6 we recall some basic facts about the Cauchy transform $\mathcal{C}$ and the Beurling transform $\mathcal{S}$, and they are applied in §2.7 to commutators of the form $\mathcal{S}b - b\mathcal{S}$, where $b \in \mathrm{BMO}(\mathbb{C})$. Banach space geometry is a central tool in this dissertation, and in §2.9 we present some of the basic definitions needed.

2.1. **Sobolev spaces.**

We first discuss the Sobolev spaces that are used in this dissertation; some of the standard references are [AF03] and [EG92]. In the context of Sobolev spaces partial derivatives are understood in the distributional sense.

Definition 2.1. The *homogeneous Sobolev space* $\dot{W}^{1,2}(\mathbb{C},\mathbb{C})$ is defined by

$$\dot{W}^{1,2}(\mathbb{C},\mathbb{C}) := \{u = u_1 + iu_2 \in L^1_{loc}(\mathbb{C},\mathbb{C})\colon \mathrm{D}u \in L^2(\mathbb{C},\mathbb{R}^{2\times 2})\},$$

and $\dot{W}^{1,2}(\mathbb{C},\mathbb{C})$ is equipped with the seminorm given by

$$\|u\|_{\dot{W}^{1,2}} := \|u_{\bar{z}}\|_{L^2}.$$

Since $\|u\|_{\dot{W}^{1,2}} = 0$ whenever u is a constant function, $\|\cdot\|_{\dot{W}^{1,2}}$ is only a seminorm on $\dot{W}^{1,2}(\mathbb{C},\mathbb{C})$. This is a slight nuisance, and one apparently natural option would be to identify mappings in $\dot{W}^{1,2}(\mathbb{C},\mathbb{C})$ that differ by a constant. Compositions $\eta \circ u$ with $\eta \in C_0^\infty(\mathbb{C},\mathbb{C})$ and $u \in \dot{W}^{1,2}(\mathbb{C},\mathbb{C})$ are, however, an important tool in this dissertation, and as a result of the identification such compositions would cease to be well-defined.

Instead of factoring out constants we introduce the subspace of functions whose integral over the unit disc vanishes and define

$$\dot{W}^{1,2}_{\mathbb{D}}(\mathbb{C},\mathbb{C}) := \left\{u \in \dot{W}^{1,2}(\mathbb{C},\mathbb{C})\colon \int_{\mathbb{D}} u = 0\right\}.$$

Here and elsewhere in this dissertation $\mathbb{D}$ is the unit disc. Note that $\dot{W}^{1,2}(\mathbb{C},\mathbb{C}) = \dot{W}^{1,2}_{\mathbb{D}}(\mathbb{C},\mathbb{C}) + \mathbb{C}$. The range of J is obviously not affected by switching the domain from $\dot{W}^{1,2}(\mathbb{C},\mathbb{C})$ to $\dot{W}^{1,2}_{\mathbb{D}}(\mathbb{C},\mathbb{C})$, and depending on the context we use whichever space is more convenient.

A proof of the following standard fact can be found at [OS12, p. 3].

Proposition 2.2. *$C_0^\infty(\mathbb{C},\mathbb{C})$ is dense in $\dot{W}^{1,2}(\mathbb{C},\mathbb{C})$.*

We also use (inhomogeneous) Sobolev spaces in domains of the complex plane.

Definition 2.3. When $\Omega \subset \mathbb{C}$ is a domain the *Sobolev space* $W^{1,2}(\Omega,\mathbb{C})$ consists of functions $u \in L^2(\Omega,\mathbb{C})$ such that $\mathrm{D}u \in L^2(\Omega,\mathbb{R}^{2\times 2})$.

For computational simplicity we use the Hilbert-Schmidt norm given by

$$|\mathrm{D}u|^2 := (\partial_x u_1)^2 + (\partial_y u_1)^2 + (\partial_x u_2)^2 + (\partial_y u_2)^2 = 2(|u_z|^2 + |u_{\bar{z}}|^2)$$

instead of the operator norm. The Sobolev space $W^{1,2}(\Omega,\mathbb{C})$ becomes a Hilbert space when equipped with the norm $\|\cdot\|_{W^{1,2}(\Omega,\mathbb{C})}$ given by

$$\|u\|_{W^{1,2}(\Omega,\mathbb{C})} := \left(\|u\|^2_{L^2(\Omega,\mathbb{C})} + \|\mathrm{D}u\|^2_{L^2(\Omega,\mathbb{R}^{2\times 2})}\right)^{\frac{1}{2}}.$$

Definition 2.4. The closure of $C_0^\infty(\Omega,\mathbb{C})$ in $W^{1,2}(\Omega,\mathbb{C})$ is denoted by $W_0^{1,2}(\Omega,\mathbb{C})$.

If $u \in W_0^{1,2}(\Omega,\mathbb{C})$, then the mapping defined by

$$U(z) := \begin{cases} u(z), & z \in \Omega, \\ 0, & z \in \mathbb{C} \setminus \Omega \end{cases}$$

belongs to $W^{1,2}(\mathbb{C},\mathbb{C})$. The converse does not hold for a general domain Ω. It does, however, hold when Ω is a Lipschitz domain.

Definition 2.5. A domain $\Omega \subset \mathbb{C}$ is said to be a *Lipschitz domain* if for every point $z \in \partial\Omega$ there exist $l > 0$ and a Lipschitz function $g\colon \mathbb{R} \to \mathbb{R}$ such that, upon rotating and relabeling the coordinate axes of $\mathbb{C}$ if necessary,

$$\Omega \cap Q(z,l) = \{w = w_1 + iw_2 \colon g(w_1) < w_2\} \cap Q(z,l),$$

where $Q(z,l)$ is the open square with center z and sidelength l.

When Ω is a Lipschitz domain, there exists, furthermore, a bounded linear operator $E\colon W^{1,2}(\Omega,\mathbb{C}) \to W^{1,2}(\mathbb{C},\mathbb{C})$ such that $Eu|_\Omega = u$ for every $u \in W^{1,2}(\Omega,\mathbb{C})$ (see [EG92, p. 135]).

2.2. **Invertibility of Sobolev mappings.** Recall that if $u \in C^1(\mathbb{C},\mathbb{C})$ has nonvanishing Jacobian at a point $z \in \mathbb{C}$, then u is locally invertible by the Inverse function theorem. The invertibility properties of Sobolev mappings are more delicate, as we discuss in this section. We denote the operator norm of a matrix $A \in \mathbb{R}^{2\times 2}$ by $\|A\|$.

Definition 2.6. Suppose $\Omega \subset \mathbb{C}$ is a bounded domain and $u \in W^{1,2}(\Omega,\mathbb{C})$. Then u is said to have *finite distortion* if

$$\|\mathrm{D}u(z)\|^2 \le K(z) J_u(z) \tag{2.1}$$

for a.e. $z \in \Omega$ and $1 \le K(z) < \infty$ for a.e. $z \in \Omega$.

Suppose now $u \in W^{1,2}(\Omega,\mathbb{C})$ has finite distortion. By (2.1), $J_u \ge 0$ a.e. in Ω. In the points where $J_u > 0$ the linear distortion K_u is defined by

$$K_u(z) := \frac{\|\mathrm{D}u(z)\|^2}{J_u(z)} = \frac{(|u_z| + |u_{\bar z}|)^2}{J_u(z)} \ge 1.$$

Since $|\mathrm{D}u|^2 = 2(|u_z|^2 + |u_{\bar z}|^2)$, it follows that

$$\frac{|\mathrm{D}u|^2}{2J_u} \le K_u \le \frac{|\mathrm{D}u|^2}{J_u}.$$

Condition (2.1) implies that $\mathrm{D}u(z) = 0$ a.e. in the critical set

$$Z_u := \{z \in \Omega \colon J_u(z) = 0\},$$

and in points of Z_u we set $K_u(z) := 1$.

When the Jacobian of a mapping $u \in W^{1,2}(\Omega,\mathbb{C})$ is positive and no higher regularity results on u are available, $K_u \in L^1(\Omega)$ is a particularly natural qualitative condition. Suppose, say, $0 < c \le J_u \le C < \infty$. Then $|\mathrm{D}u|^2/2C \le K_u \le |\mathrm{D}u|^2/c$ and so

$$\int_\Omega K_u \lesssim_c \int_\Omega |\mathrm{D}u|^2.$$

However, if $\mathrm{D}u$ does not enjoy higher integrability, the linear distortion K_u cannot belong to $L^{1+\epsilon}(\Omega)$ for any $\epsilon > 0$.

In [IS93] T. Iwaniec and V. Svérak proved the following Stoilow-type factorization result for mappings with integrable distortion.

Theorem 2.7. *Let $\Omega \subset \mathbb{C}$ be a bounded domain and suppose $u \in W^{1,2}(\Omega, \mathbb{C})$ satisfies $J_u \geq 0$ a.e. and $K_u \in L^1(\Omega)$. Then there exists another domain $\Omega' \subset \mathbb{C}$, a homeomorphism $g\colon \Omega \to \Omega'$ and an analytic function $\phi\colon \Omega' \to \mathbb{C}$ such that $u = \phi \circ g$.*

Theorem 2.7 allows us to conclude that a mapping with integrable distortion is a local homeomorphism a.e.

Corollary 2.8. *Let Ω be a bounded domain in $\mathbb{C}$ and suppose a non-constant mapping $u \in W^{1,2}(\Omega, \mathbb{C})$ satisfies $J_u \geq 0$ and $K_u \in L^1(\Omega)$. Then u is a local homeomorphism in $\Omega \setminus S$, where $S \subset \Omega$ is discrete.*

Locally, outside a discrete set, Corollary 2.8 can be used to assume that u is a homeomorphism. Then the following result of K. Astala, T. Iwaniec, G. Martin and J. Onninen (see [AIMO05, Theorem 9.1]) gives important information about the local inverse.

Theorem 2.9. *Let $\Omega \subset \mathbb{C}$ be a bounded domain and suppose a homeomorphism $u \in W^{1,2}(\Omega, u(\Omega))$ satisfies $J_u \geq 0$ a.e. and $K_u \in L^1(\Omega)$. Then $u^{-1} \in W^{1,2}(u(\Omega), \Omega)$ and*

$$\int_{u(\Omega)} \| \operatorname{D} u^{-1} \|^2 = \int_\Omega K_u.$$

2.3. **The Hardy space $\mathcal{H}^1(\mathbb{C})$.** Another function space essential to this dissertation is the Hardy space $\mathcal{H}^1(\mathbb{C})$ defined originally by E. Stein and G. Weiss. In this section we give a definition of $\mathcal{H}^1(\mathbb{C})$ and briefly discuss the atomic decomposition of $\mathcal{H}^1(\mathbb{C})$. The results and their proofs can be found in the books [Ste93] and [Gra04].

We fix, for the remainder of this dissertation, a smooth function $\Phi : \mathbb{C} \to \mathbb{R}$ that satisfies $\operatorname{supp}(\Phi) \subset \mathbb{D}$ and $\int_\mathbb{C} \Phi = 1$. When $t > 0$, the function $\Phi_t \in C_0^\infty(\mathbb{C})$ is defined by

$$\Phi_t(z) := \frac{1}{t^2} \Phi\left(\frac{z}{t}\right).$$

A change of variables yields $\int_\mathbb{C} \Phi_t = 1$ for every $t > 0$. The *smooth maximal function* of a tempered distribution $h \in \mathcal{S}'(\mathbb{C})$ is given by

$$\mathcal{M}h(z) := \sup_{t>0} |h * \Phi_t(z)|.$$

Definition 2.10. The *Hardy space* $\mathcal{H}^1(\mathbb{C})$ is defined by

$$\mathcal{H}^1(\mathbb{C}) := \{h \in \mathcal{S}'(\mathbb{C})\colon \mathcal{M}h \in L^1(\mathbb{C})\}.$$

We endow $\mathcal{H}^1(\mathbb{C})$ with the norm

$$\|h\|_{\mathcal{H}^1} := \|\mathcal{M}h\|_{L^1},$$

and as a result $\mathcal{H}^1(\mathbb{C})$ becomes a Banach space (see [Gra04, Proposition 6.4.10]). There exists a plethora of equivalent norms that are used in $\mathcal{H}^1(\mathbb{C})$ (see [Gra04, Theorem 6.4.4]). In particular, in the definition of the smooth maximal function $\mathcal{M}h$ we can replace Φ by any $\Psi \in \mathcal{S}(\mathbb{C})$ that satisfies $\int_\mathbb{C} \Psi \neq 0$ and thereby get an equivalent norm for $\mathcal{H}^1(\mathbb{C})$. In §2.8 we recall the definition of another norm that is particularly well-suited to the study of the Jacobian equation.

Members of $\mathcal{H}^1(\mathbb{C})$ are integrable. Furthermore, if $h \in \mathcal{H}^1(\mathbb{C})$, then $\int_{\mathbb{C}} h = 0$. The converse does not hold, but the following proposition gives a partial converse. The result can be proved by adapting the argument from [Ste93, p. 106].

Proposition 2.11. *Suppose $1 < p < \infty$ and a function $h \in L^p(\mathbb{C})$ is supported on a square $Q \subset \mathbb{C}$ and satisfies $\int_{\mathbb{C}} h = 0$. Then $h \in \mathcal{H}^1(\mathbb{C})$ and h satisfies the inequality*

$$\|h\|_{\mathcal{H}^1} \lesssim_p |Q|^{\frac{1}{p'}} \|h\|_{L^p(Q)}.$$

One of the central theorems on $\mathcal{H}^1(\mathbb{C})$ is the decomposition of $\mathcal{H}^1$ functions into so-called $\mathcal{H}^1$ atoms which are defined as follows.

Definition 2.12. A measurable function $a : \mathbb{C} \to \mathbb{R}$ is called an $\mathcal{H}^1$ *atom* if it satisfies the following properties:
(i) a is supported in a square $Q \subset \mathbb{C}$,
(ii) $\|a\|_{L^\infty} \leq 1/|Q|$,
(iii) $\int_{\mathbb{C}} a = 0$.
We also call a a Q-atom.

Suppose now a is a Q-atom. Then Proposition 2.11, Hölder's inequality and condition (iii) in Definition 2.12 give

$$\|a\|_{\mathcal{H}^1} \lesssim |Q|^{\frac{1}{2}} \left(\int_Q |a|^2 \right)^{\frac{1}{2}} \leq |Q|^{\frac{1}{2}} \left(|Q| \|a\|_{L^\infty}^2 \right)^{\frac{1}{2}} \leq 1,$$

and so a belongs to $\mathcal{H}^1(\mathbb{C})$ and has a uniformly bounded $\mathcal{H}^1$ norm.

The following fundamental result, the atomic decomposition of Hardy spaces, was proved by R.R. Coifman in one dimension and generalized by R.H. Latter to higher dimensions. We only state the atomic decomposition theorem in the case relevant to us, namely $\mathcal{H}^1(\mathbb{C})$ (see e.g. [Ste93, p. 107]).

Theorem 2.13. *Let $h \in \mathcal{H}^1(\mathbb{C})$. Then there exist $\lambda_j \in \mathbb{R}$ and $\mathcal{H}^1$ atoms a_j such that*

$$\lim_{n\to\infty} \left\| h - \sum_{j=1}^{n} \lambda_j a_j \right\|_{\mathcal{H}^1} = 0,$$

$$\sum_{j=1}^{\infty} |\lambda_j| \lesssim \|h\|_{\mathcal{H}^1}.$$

The decomposition given in Theorem 2.13 is not unique, neither is it linear with respect to h. However, $\mathcal{H}^1$ atoms are so much easier to use than general $\mathcal{H}^1$ functions that the atomic decomposition has found extensive use in analysis. The following proposition presents one application of the atomic decomposition of $\mathcal{H}^1(\mathbb{C})$.

Proposition 2.14. *The set*

$$C_\bullet^\infty(\mathbb{C}) := \left\{ \varphi \in C_0^\infty(\mathbb{C}) \colon \int_{\mathbb{C}} \varphi = 0 \right\}$$

is a dense subspace of $\mathcal{H}^1(\mathbb{C})$.

We briefly sketch the idea of the proof. Note first that by Theorem 2.13, finite linear combinations of $\mathcal{H}^1$ atoms are dense in $\mathcal{H}^1(\mathbb{C})$. When h is such a linear combination, the mollified functions $h * \Phi_t$, $t > 0$, belong to $C^\infty_\bullet(\mathbb{C})$. Proposition 2.11 can then be used to approximate h by the functions $h * \Phi_t$.

Another basic property of $\mathcal{H}^1(\mathbb{C})$ is presented in the following result. It turns out to be crucial to us, as we use Banach space geometry as one of our main tools.

Proposition 2.15. *The Hardy space $\mathcal{H}^1(\mathbb{C})$ is separable.*

Proof. When $k \in \mathbb{N}$, denote

$$L^2_0(\mathbb{D}(0,k)) := \left\{ f \in L^2(\mathbb{C})\colon \operatorname{supp}(f) \subset \overline{\mathbb{D}(0,k)} \text{ and } \int_{\mathbb{C}} f = 0 \right\}.$$

As a subspace of $L^2(\mathbb{C})$, the space $L^2_0(\mathbb{D}(0,k))$ is separable. The claim follows by choosing a dense countable set in $L^2_0(\mathbb{D}(0,k))$ for each $k \in \mathbb{N}$ and using Propositions 2.11 and 2.14. □

We also need a Hardy-type space on bounded Lipschitz domains of $\mathbb{C}$; [JSW84], [Miy90], [CKS93], [Cha94], [CDS99] and [AR03] are a few of the main articles on this topic. When $\Omega \subset \mathbb{C}$ is such a domain, the Hardy space $\mathcal{H}^1_z(\Omega)$ consists of functions whose zero extension to $\mathbb{C} \setminus \Omega$ belongs to $\mathcal{H}^1(\mathbb{C})$. More precisely:

Definition 2.16. A function $h \in L^1_{loc}(\Omega)$ belongs to $\mathcal{H}^1_z(\Omega)$ if the extension $H\colon \mathbb{C} \to \mathbb{R}$,

$$H(z) := \begin{cases} h(z), & z \in \Omega, \\ 0, & z \in \mathbb{C} \setminus \Omega, \end{cases}$$

belongs to $\mathcal{H}^1(\mathbb{C})$.

The norm defined by $\|h\|_{\mathcal{H}^1_z(\Omega)} := \|H\|_{\mathcal{H}^1}$ makes $\mathcal{H}^1_z(\Omega)$ a Banach space.

2.4. **Jacobians of Sobolev mappings.** As we mentioned in the Introduction, R.R. Coifman, P.-L. Lions, Y. Meyer and S. Semmes proved in [CLMS93] that Jacobians of mappings in $\dot{W}^{1,2}(\mathbb{C},\mathbb{C})$ belong to $\mathcal{H}^1(\mathbb{C})$. The main motivation behind the work done in [CLMS93] was the following result of S. Müller (see [Mül89, Theorem 1]). Müller's theorem and the other results mentioned in this section were proved in $\mathbb{R}^n$, $n \geq 2$, but we formulate them only in the case relevant to us, the complex plane.

Theorem 2.17. *Let $\Omega \subset \mathbb{C}$ be open and suppose $u \in W^{1,2}(\Omega,\mathbb{C})$. If $J_u \geq 0$ a.e., then $J_u \log(2 + J_u) \in L^1_{loc}(\Omega)$.*

On the other hand, it follows from an earlier result of E.M. Stein (see [Ste69, Theorem 1]) that if $h \in \mathcal{H}^1(\mathbb{C})$ satisfies $h \geq 0$ in an open set $\Omega \subset \mathbb{C}$, then $h \log(2+h) \in L^1_{loc}(\Omega)$. We now restate the theorem of R.R. Coifman, P.-L. Lions, Y. Meyer and S. Semmes.

Theorem 2.18. *Let $u = u_1 + iu_2 \in \dot{W}^{1,2}(\mathbb{C},\mathbb{C})$. Then the Jacobian J_u is in $\mathcal{H}^1(\mathbb{C})$ and satisfies the estimate $\|J_u\|_{\mathcal{H}^1} \lesssim \|\nabla u_1\|_{L^2} \|\nabla u_2\|_{L^2}$.*

Theorem 2.18 has the following implication.

Corollary 2.19. *The operator $J\colon \dot{W}^{1,2}(\mathbb{C},\mathbb{C}) \to \mathcal{H}^1(\mathbb{C})$ is locally Lipschitz continuous.*

Proof. Let $u, v \in \dot{W}^{1,2}(\mathbb{C}, \mathbb{C})$. We write

$$J_u - J_v = J_{u_1+i(u_2-v_2)} + J_{u_1-v_1+iv_2}$$

and use Theorem 2.18 to estimate

$$\begin{aligned}
\|J_u - J_v\|_{\mathcal{H}^1} &\leq \|J_{u_1+i(u_2-v_2)}\|_{\mathcal{H}^1} + \|J_{u_1-v_1+iv_2}\|_{\mathcal{H}^1} \\
&\lesssim \|\nabla u_1\|_{L^2}\|\nabla(u_2 - v_2)\|_{L^2} + \|\nabla(u_1 - v_1)\|_{L^2}\|\nabla v_2\|_{L^2} \\
&\lesssim (\|\nabla u_1\|_{L^2} + \|\nabla v_2\|_{L^2})\|u - v\|_{\dot{W}^{1,2}}.
\end{aligned}$$

This proves the local Lipschitz continuity of J. □

Coifman, Lions, Meyer and Semmes also proved the following Jacobian decomposition theorem. A quick way to deduce Theorem 2.20 from the atomic decomposition of $\mathcal{H}^1(\mathbb{C})$ is shown in [LYS05].

Theorem 2.20. *Suppose $h \in \mathcal{H}^1(\mathbb{C})$. Then there exist $\lambda_j \in \mathbb{R}$ and $u^j \in \dot{W}^{1,2}(\mathbb{C}, \mathbb{C})$ such that*

$$h = \sum_{j=1}^{\infty} \lambda_j J_{u^j}$$

(convergence in the norm $\|\cdot\|_{\mathcal{H}^1}$),

$$\sum_{j=1}^{\infty} |\lambda_j| \lesssim \|h\|_{\mathcal{H}^1}$$

and

$$\int_{\mathbb{C}} |\operatorname{D} u^j|^2 \leq 1 \quad \textit{for all } j \in \mathbb{N}.$$

We shall also use the following classical result which says that the Jacobian is a null Lagrangian (see [AIM09, Corollary 2.9.3]).

Proposition 2.21. *Let $\Omega \subset \mathbb{C}$ be a domain and suppose $u, v \in W^{1,2}(\Omega, \mathbb{C})$ and $u - v \in W_0^{1,2}(\Omega, \mathbb{C})$. Then*

$$\int_\Omega J_u = \int_\Omega J_v.$$

2.5. $\mathrm{VMO}(\mathbb{C})$ **and** $\mathrm{BMO}(\mathbb{C})$**.** By a famous theorem of C. Fefferman, the space of functions of bounded mean oscillation is the dual space of $\mathcal{H}^1(\mathbb{C})$. In this section we discuss Fefferman's theorem and the later result of R.R. Coifman and G. Weiss that says that $\mathcal{H}^1(\mathbb{C})$ is the dual of the space of functions of vanishing mean oscillation. As in the case of $\mathcal{H}^1(\mathbb{C})$, [Ste93] and [Gra04] are some of the standard references.

When $b \in L^1_{loc}(\mathbb{C})$ and $Q \subset \mathbb{C}$ is a square, the average of b over Q is denoted by $b_Q := \fint_Q b := |Q|^{-1} \int_Q b$.

Definition 2.22. A function $b \in L^1_{loc}(\mathbb{C})$ belongs to $\mathrm{BMO}(\mathbb{C})$, the space of functions of *bounded mean oscillation*, if there exists a constant $A \geq 0$ such that

$$\fint_Q |b - b_Q| \leq A \quad \text{for every square } Q \subset \mathbb{C}. \tag{2.2}$$

The smallest constant for which (2.2) holds is denoted by $\|b\|_{\mathrm{BMO}}$.

To be precise, in BMO($\mathbb{C}$) functions that differ by a constant are identified. Then $\|\cdot\|_{\mathrm{BMO}}$ becomes a norm with respect to which BMO($\mathbb{C}$) is a Banach space. One of the central properties of BMO($\mathbb{C}$) is the following higher integrability theorem of F. John and L. Nirenberg.

Theorem 2.23. *Let $b \in \mathrm{BMO}(\mathbb{C})$. If $1 < p < \infty$, then $b \in L^p_{loc}(\mathbb{C})$ and b satisfies the estimate*

$$\left(\frac{1}{|Q|}\int_Q |b - b_Q|^p\right)^{\frac{1}{p}} \lesssim_p \|b\|_{\mathrm{BMO}}$$

for every square $Q \subset \mathbb{C}$.

The duality theorem of C. Fefferman is another fundamental result on BMO($\mathbb{C}$). We denote the space of finite linear combinations of $\mathcal{H}^1$ atoms by $\mathcal{H}^1_a(\mathbb{C})$. By Theorem 2.13, $\mathcal{H}^1_a(\mathbb{C})$ is dense in $\mathcal{H}^1(\mathbb{C})$. Note that if $h \in \mathcal{H}^1_a(\mathbb{C})$, then h is bounded and compactly supported and therefore $bh \in L^1(\mathbb{C})$ for every $b \in \mathrm{BMO}(\mathbb{C})$.

Theorem 2.24. *Suppose $b \in \mathrm{BMO}(\mathbb{C})$. Then the linear functional $\ell\colon \mathcal{H}^1_a(\mathbb{C}) \to \mathbb{R}$ defined by the Lebesgue integral*

$$\langle \ell, h\rangle := \int_{\mathbb{C}} bh \tag{2.3}$$

has a unique bounded extension to $\mathcal{H}^1(\mathbb{C})$ and satisfies the estimate

$$\|\ell\|_{(\mathcal{H}^1)^*} := \sup_{\|h\|_{\mathcal{H}^1}\le 1} \langle \ell, h\rangle \lesssim \|b\|_{\mathrm{BMO}}. \tag{2.4}$$

Conversely, for every continuous linear functional ℓ on $\mathcal{H}^1(\mathbb{C})$ there exists $b \in \mathrm{BMO}(\mathbb{C})$ such that (2.3) holds for all $h \in \mathcal{H}^1_a(\mathbb{C})$ and

$$\|b\|_{\mathrm{BMO}} \lesssim \|\ell\|_{(\mathcal{H}^1)^*}.$$

The element $b \in \mathrm{BMO}(\mathbb{C})$ corresponding to ℓ is unique.

As is customary, we identify the linear functional ℓ with the function $b \in \mathrm{BMO}(\mathbb{C})$. Following [BIJZ07] we denote the dual pairing by

$$\int_{\mathbb{C}}^* bh := \langle b, h\rangle = \lim_{n\to\infty}\int_{\mathbb{C}} b\sum_{j=1}^n \lambda_j a_j, \tag{2.5}$$

where $\sum_{j=1}^\infty \lambda_j a_j$ is an atomic decomposition of h generated by Theorem 2.13. The value of $\int_{\mathbb{C}}^* bh$ does not depend on the choice of the decomposition.

The pointwise product of $b \in \mathrm{BMO}(\mathbb{C})$ and $h \in \mathcal{H}^1(\mathbb{C})$ neet not be integrable, even locally (see [Ste93, p. 178]). Therefore the dual pairing of b and h needs to be interpreted in some manner, for instance via (2.5). Different ways to give $\int_{\mathbb{C}}^* bh$ a meaning are discussed in [BIJZ07]. However, if $bh \in L^1(\mathbb{C})$, then $\int_{\mathbb{C}}^* bh = \int_{\mathbb{C}} bh$, and this provides a justification for the suggestive notation $\int_{\mathbb{C}}^* bh$.

Not only is BMO($\mathbb{C}$) the dual of $\mathcal{H}^1(\mathbb{C})$, R.R. Coifman and G. Weiss proved (see [CW77, Theorem 4.2]) that $\mathcal{H}^1(\mathbb{C})$ is the dual of a subspace of BMO($\mathbb{C}$) called VMO($\mathbb{C}$), the space of functions of *vanishing mean oscillation*. There exist two different and non-equivalent definitions of VMO($\mathbb{C}$) in the literature, but in this dissertation the following one is used.

Definition 2.25. VMO($\mathbb{C}$) is the closure of $C_0^\infty(\mathbb{C})$ in BMO($\mathbb{C}$).

It turns out that VMO functions have useful properties that general BMO functions lack. One example is discussed in §2.7; the commutator of a function $b \in \mathrm{BMO}(\mathbb{C})$ with the Beurling transform is a bounded operator in $L^2(\mathbb{C}, \mathbb{C})$, but if b belongs to $\mathrm{VMO}(\mathbb{C})$, the commutator is a compact operator.

2.6. **The Cauchy transform and the Beurling transform.** Jacobians of mappings in $\dot{W}^{1,2}(\mathbb{C}, \mathbb{C})$ can often be studied more conveniently by switching to the setting of $L^2(\mathbb{C}, \mathbb{C})$ via the Cauchy transform and the Beurling transform. Two of the principal reasons for this are the relative ease of using duality arguments in $L^2(\mathbb{C}, \mathbb{C})$ and the nice operator theoretical properties of the Beurling transform. For proofs of the facts mentioned in this section we refer to [AIM09].

Recall that we defined $\|u\|_{\dot{W}^{1,2}} := \|u_{\bar{z}}\|_{L^2}$ for all $u \in \dot{W}^{1,2}(\mathbb{C}, \mathbb{C})$. The operator $u \mapsto u_{\bar{z}} \colon \dot{W}^{1,2}(\mathbb{C}, \mathbb{C}) \to L^2(\mathbb{C}, \mathbb{C})$ is thus an isometry. It has a right inverse, the Cauchy transform $\mathcal{C} : L^2(\mathbb{C}, \mathbb{C}) \to \dot{W}^{1,2}(\mathbb{C}, \mathbb{C})$ defined for $\phi \in C_0^\infty(\mathbb{C}, \mathbb{C})$ by the integral

$$\mathcal{C}\phi(z) := \frac{1}{\pi} \int_{\mathbb{C}} \frac{\phi(w)}{z - w} \, dw$$

which converges for every $z \in \mathbb{C}$.

For the extension of $\mathcal{C}$ to a bounded linear mapping from $L^2(\mathbb{C}, \mathbb{C})$ into $\mathrm{VMO}(\mathbb{C}, \mathbb{C})$ see [AIM09, Theorem 4.3.9]. This extension is only defined up to an additive constant, but every representative of $\mathcal{C}f \in \mathrm{VMO}(\mathbb{C}, \mathbb{C})$ belongs to $\dot{W}^{1,2}(\mathbb{C}, \mathbb{C})$ and satisfies $(\mathcal{C}f)_{\bar{z}} = f$. We thus have the following crucial result.

Lemma 2.26. *Suppose $f \in L^2(\mathbb{C}, \mathbb{C})$. Then there exists $u \in \dot{W}^{1,2}(\mathbb{C}, \mathbb{C})$ such that*

$$u_{\bar{z}} = f.$$

Another integral transform we need is the Beurling transform $\mathcal{S}$ given, for $f \in C_0^\infty(\mathbb{C}, \mathbb{C})$, by the Cauchy principal value integral

$$\mathcal{S}f(z) := -\frac{1}{\pi} \lim_{\epsilon \searrow 0} \int_{\mathbb{C} \setminus B(0,\epsilon)} \frac{f(w)}{(z - w)^2} \, dw \tag{2.6}$$

which converges for every $z \in \mathbb{C}$. The formula

$$\|\mathcal{S}f\|_{L^2} = \|f\|_{L^2} \tag{2.7}$$

holds for every $f \in C_0^\infty(\mathbb{C}, \mathbb{C})$, and therefore the Beurling transform can be extended to an isometry $\mathcal{S} : L^2(\mathbb{C}, \mathbb{C}) \to L^2(\mathbb{C}, \mathbb{C})$. Identity (2.6) holds at a.e. $z \in \mathbb{C}$ for the extended operator. The operator $\mathcal{S}$ is symmetric in the sense that $\int_{\mathbb{C}} f \mathcal{S}g = \int_{\mathbb{C}} (\mathcal{S}f) g$ for all $f, g \in L^2(\mathbb{C}, \mathbb{C})$. Furthermore, every $f \in L^2(\mathbb{C}, \mathbb{C})$ satisfies $\bar{f} = \mathcal{S}\overline{\mathcal{S}f}$.

When $u \in \dot{W}^{1,2}(\mathbb{C}, \mathbb{C})$, the Beurling transform intertwines the Wirtinger derivatives of u via the formula $\mathcal{S}u_{\bar{z}} = u_z$. Consequently,

$$J_u = |u_z|^2 - |u_{\bar{z}}|^2 = |\mathcal{S}u_{\bar{z}}|^2 - |u_{\bar{z}}|^2.$$

On the other hand, if $f \in L^2(\mathbb{C}, \mathbb{C})$, then Lemma 2.26 implies that $|\mathcal{S}f|^2 - |f|^2 = J_u$ for some $u \in \dot{W}^{1,2}(\mathbb{C}, \mathbb{C})$. The question about the surjectivity of $J \colon \dot{W}^{1,2}(\mathbb{C}, \mathbb{C}) \to \mathcal{H}^1(\mathbb{C})$ can thus be posed equivalently by asking whether the operator

$$f \mapsto |\mathcal{S}f|^2 - |f|^2 : L^2(\mathbb{C}, \mathbb{C}) \to \mathcal{H}^1(\mathbb{C})$$

maps $L^2(\mathbb{C}, \mathbb{C})$ onto $\mathcal{H}^1(\mathbb{C})$.

2.7. **Commutators.** Commutators of the form $\mathcal{S}b - b\mathcal{S}$, where $b \in \mathrm{BMO}(\mathbb{C})$, turn out to be very useful in studying the Jacobian equation. In this section we define the commutators and mention some of their basic properties. Proofs can be found e.g. in [AIM09].

When $b \in \mathrm{BMO}(\mathbb{C})$, Theorem 2.23 says that $b \in L^2_{loc}(\mathbb{C})$. Hence, given $g \in C_0^\infty(\mathbb{C},\mathbb{C})$ the product bg belongs to $L^2(\mathbb{C},\mathbb{C})$, and therefore $\mathcal{S}(bg) \in L^2(\mathbb{C},\mathbb{C})$. Since the functions b and $\mathcal{S}g$ both belong to $L^2_{loc}(\mathbb{C},\mathbb{C})$, we also have $b\mathcal{S}g \in L^1_{loc}(\mathbb{C},\mathbb{C})$. Thus

$$\mathcal{S}(bg) - b\mathcal{S}g \in L^1_{loc}(\mathbb{C},\mathbb{C}).$$

The commutator $\mathcal{S}b - b\mathcal{S}$ is therefore a well-defined mapping from $C_0^\infty(\mathbb{C},\mathbb{C})$ to $L^1_{loc}(\mathbb{C},\mathbb{C})$. Much more can be said about it; as a special case of a famous result of R.R. Coifman, R. Rochberg and G. Weiss the commutator $\mathcal{S}b - b\mathcal{S}$ is bounded on $L^2(\mathbb{C},\mathbb{C})$.

Theorem 2.27. *When $b \in \mathrm{BMO}(\mathbb{C})$ and $g \in C_0^\infty(\mathbb{C},\mathbb{C})$, the commutator $(\mathcal{S}b - b\mathcal{S})g$ belongs to $L^2(\mathbb{C},\mathbb{C})$ and the following inequality holds:*

$$\|(\mathcal{S}b - b\mathcal{S})g\|_{L^2} \lesssim \|b\|_{\mathrm{BMO}}\|g\|_{L^2}. \tag{2.8}$$

As a consequence the linear operator $g \mapsto (\mathcal{S}b - b\mathcal{S})g$ has a bounded extension that satisfies (2.8) for every $g \in L^2(\mathbb{C},\mathbb{C})$.

The following important result is due to A. Uchiyama.

Theorem 2.28. *When $b \in \mathrm{VMO}(\mathbb{C})$, the commutator*

$$\mathcal{S}b - b\mathcal{S} \colon L^2(\mathbb{C},\mathbb{C}) \to L^2(\mathbb{C},\mathbb{C})$$

is a compact operator.

Commutators are related to Jacobians via the identity given in the following result from [AIM09, p. 547].

Lemma 2.29. *Suppose $b \in \mathrm{BMO}(\mathbb{C})$ and $f \in L^2(\mathbb{C},\mathbb{C})$. Then*

$$\int_{\mathbb{C}}^{*} b(|\mathcal{S}f|^2 - |f|^2) = \int_{\mathbb{C}} f(\mathcal{S}b - b\mathcal{S})\overline{\mathcal{S}f}. \tag{2.9}$$

Proof. We prove (2.9) by an approximation argument. By Lemma 2.26 there exists $u \in \dot{W}^{1,2}(\mathbb{C},\mathbb{C})$ such that $u_{\bar{z}} = f$. Next use Proposition 2.2 to select a sequence of mappings $\phi^j \in C_0^\infty(\mathbb{C},\mathbb{C})$ such that $\lim_{j\to\infty} \|\phi^j - u\|_{\dot{W}^{1,2}} = 0$. By Corollary 2.19,

$$\|J_{\phi^j} - J_u\|_{\mathcal{H}^1} \to 0. \tag{2.10}$$

Fix $j \in \mathbb{N}$, denote

$$f^j := \phi^j_{\bar{z}} \in C_0^\infty(\mathbb{C},\mathbb{C})$$

and note that $\mathcal{S}f^j = \phi^j_z \in C_0^\infty(\mathbb{C},\mathbb{C})$. Thus $b\overline{\mathcal{S}f^j} \in L^2(\mathbb{C},\mathbb{C})$. Since $\mathcal{S}$ is symmetric in $L^2(\mathbb{C},\mathbb{C})$ and $\overline{f^j} = \mathcal{S}\overline{\mathcal{S}f^j}$, we may compute

$$\begin{aligned}
\int_{\mathbb{C}}^{*} b(|\mathcal{S}f^j|^2 - |f^j|^2) &= \int_{\mathbb{C}} b\mathcal{S}f^j\overline{\mathcal{S}f^j} - \int_{\mathbb{C}} bf^j\overline{f^j} \\
&= \int_{\mathbb{C}} f^j\mathcal{S}(b\overline{\mathcal{S}f^j}) - \int_{\mathbb{C}} f^j b\mathcal{S}\overline{\mathcal{S}f^j} \\
&= \int_{\mathbb{C}} f^j(\mathcal{S}b - b\mathcal{S})\overline{\mathcal{S}f^j}.
\end{aligned}$$

By (2.10),

$$\int_{\mathbb{C}}^{*} b(|\mathcal{S}f^j|^2 - |f^j|^2) = \int_{\mathbb{C}}^{*} bJ_{\phi^j} \to \int_{\mathbb{C}}^{*} bJ_u = \int_{\mathbb{C}}^{*} b(|\mathcal{S}f|^2 - |f|^2).$$

On the other hand, since $f^j \to f$ in $L^2(\mathbb{C},\mathbb{C})$, Theorem 2.27 and formula (2.7) imply that $\lim_{j\to\infty} \|(\mathcal{S}b - b\mathcal{S})\overline{\mathcal{S}(f^j - f)}\|_{L^2} = 0$, and so

$$\int_{\mathbb{C}} f^j(\mathcal{S}b - b\mathcal{S})\overline{\mathcal{S}f^j} \to \int_{\mathbb{C}} f(\mathcal{S}b - b\mathcal{S})\overline{\mathcal{S}f}.$$

The proof of (2.9) is complete. $\square$

Lemma 2.29 yields the following result. When stated in terms of Jacobians, Proposition 2.30 was (essentially) proved by R. Caccioppoli and generalized to higher dimensions by C.B. Morrey.

Proposition 2.30. *Suppose $f^j \rightharpoonup f$ in $L^2(\mathbb{C},\mathbb{C})$. Then $|\mathcal{S}f^j|^2 - |f^j|^2 \overset{*}{\rightharpoonup} |\mathcal{S}f|^2 - |f|^2$ in $\mathcal{H}^1(\mathbb{C})$.*

Proof. Since $f^j \rightharpoonup f$ and $\mathcal{S}\colon L^2(\mathbb{C},\mathbb{C}) \to L^2(\mathbb{C},\mathbb{C})$ is a bounded linear operator, we have

$$\overline{\mathcal{S}f^j} \rightharpoonup \overline{\mathcal{S}f}. \tag{2.11}$$

Fix $b \in \mathrm{VMO}(\mathbb{C})$. By Theorem 2.28, $\mathcal{S}b - b\mathcal{S}\colon L^2(\mathbb{C},\mathbb{C}) \to L^2(\mathbb{C},\mathbb{C})$ is a compact operator, thus, by (2.11), $(\mathcal{S}b - b\mathcal{S})\overline{\mathcal{S}f^j} \to (\mathcal{S}b - b\mathcal{S})\overline{\mathcal{S}f}$ in $L^2(\mathbb{C},\mathbb{C})$. Finally use Lemma 2.29 to write

$$\int_{\mathbb{C}}^{*} b(|\mathcal{S}f^j|^2 - |f^j|^2) = \int_{\mathbb{C}} f^j(\mathcal{S}b - b\mathcal{S})\overline{\mathcal{S}f^j} \to \int_{\mathbb{C}} f(\mathcal{S}b - b\mathcal{S})\overline{\mathcal{S}f} = \int_{\mathbb{C}}^{*} b(|\mathcal{S}f|^2 - |f|^2).$$

$\square$

2.8. **An equivalent norm in** $\mathrm{BMO}(\mathbb{C})$**.** When R.R. Coifman, P.-L. Lions, Y. Meyer and S. Semmes proved the Jacobian decomposition of Theorem 2.20, they made use of an equivalent norm in $\mathrm{BMO}(\mathbb{C})$. We give the norm another characterization via commutators and also give an alternative proof of the equivalence of the norm and the standard norm $\|\cdot\|_{\mathrm{BMO}}$.

Definition 2.31. When $b \in \mathrm{BMO}(\mathbb{C})$, we define

$$\|b\|_{\mathrm{BMO}_{\mathcal{S}}} := \sup_{\|f\|_{L^2}=1} \int_{\mathbb{C}}^{*} b(|\mathcal{S}f|^2 - |f|^2). \tag{2.12}$$

It is straightforward to check that $\|\cdot\|_{\mathrm{BMO}_{\mathcal{S}}}$ is, indeed, a norm. The principal aim of this chapter is to give a proof of the following result from [CLMS93].

Theorem 2.32. *Suppose $b \in \mathrm{BMO}(\mathbb{C})$. Then*

$$\|b\|_{\mathrm{BMO}} \lesssim \|b\|_{\mathrm{BMO}_{\mathcal{S}}} \lesssim \|b\|_{\mathrm{BMO}}.$$

The key to proving Theorem 2.32 is identity (2.9) given in Lemma 2.29. The following definition is motivated by (2.9).

Definition 2.33. When $b \in \mathrm{BMO}(\mathbb{C})$, the operator $K_b\colon L^2(\mathbb{C},\mathbb{C}) \to L^2(\mathbb{C},\mathbb{C})$ is defined by

$$K_b f := \overline{(\mathcal{S}b - b\mathcal{S})\overline{\mathcal{S}f}}. \tag{2.13}$$

The following result collects some of the basic properties of K_b.

Lemma 2.34. *Let $b \in \mathrm{BMO}(\mathbb{C})$. Then*

$$\int_{\mathbb{C}}^{*} b(|\mathcal{S}f|^2 - |f|^2) = \int_{\mathbb{C}} f\overline{K_b f} \tag{2.14}$$

for every $f \in L^2(\mathbb{C},\mathbb{C})$ and $K_b\colon L^2(\mathbb{C},\mathbb{C}) \to L^2(\mathbb{C},\mathbb{C})$ is self-adjoint. Furthermore, if $b \in \mathrm{VMO}(\mathbb{C})$, then K_b is compact.

Proof. Identity (2.14) was proved in Lemma 2.29. Since $\int_{\mathbb{C}} f\overline{K_b f}$ is real-valued for every $f \in L^2(\mathbb{C},\mathbb{C})$, the operator K_b is self-adjoint (see [Con90, Proposition II.2.12]). If $b \in \mathrm{VMO}(\mathbb{C})$, then $\mathcal{S}b - b\mathcal{S}\colon L^2(\mathbb{C},\mathbb{C}) \to L^2(\mathbb{C},\mathbb{C})$ is a compact operator by Theorem 2.28. As a consequence, K_b is compact as well. □

As mentioned in the Introduction, when $b \in \mathrm{BMO}(\mathbb{C})$, the norm $\|b\|_{\mathrm{BMO}_\mathcal{S}}$ turns out to be equal to the operator norm $\|\mathcal{S}b - b\mathcal{S}\|_{L^2\to L^2}$.

Proposition 2.35. *Let $b \in \mathrm{BMO}(\mathbb{C})$. Then*

$$\|b\|_{\mathrm{BMO}_\mathcal{S}} = \|K_b\|_{L^2\to L^2} = \|\mathcal{S}b - b\mathcal{S}\|_{L^2\to L^2}.$$

Proof. Since $f \mapsto \overline{\mathcal{S}f}\colon L^2(\mathbb{C},\mathbb{C}) \to L^2(\mathbb{C},\mathbb{C})$ is an isometric isomorphism, we have

$$\|\mathcal{S}b - b\mathcal{S}\|_{L^2\to L^2} = \|K_b\|_{L^2\to L^2}.$$

Furthermore, since $K_b\colon L^2(\mathbb{C},\mathbb{C}) \to L^2(\mathbb{C},\mathbb{C})$ is self-adjoint, its operator norm equals its numerical radius, that is,

$$\|K_b\|_{L^2\to L^2} = \sup_{\|f\|_{L^2}=1} \left| \int_{\mathbb{C}} f\overline{K_b f} \right|$$

(see [Con90, Proposition II.2.13]). By (2.14),

$$\sup_{\|f\|_{L^2}=1} \left| \int_{\mathbb{C}} f\overline{K_b f} \right| = \sup_{\|f\|_{L^2}=1} \int_{\mathbb{C}}^{*} b(|\mathcal{S}f|^2 - |f|^2) = \|b\|_{\mathrm{BMO}_\mathcal{S}}.$$

□

When $b \in \mathrm{BMO}(\mathbb{C})$, Theorem 2.27 and Proposition 2.35 imply that $\|b\|_{\mathrm{BMO}_\mathcal{S}} \lesssim \|b\|_{\mathrm{BMO}}$. The converse inequality will also be a consequence of Proposition 2.35. The estimate $\|b\|_{\mathrm{BMO}} \lesssim \|\mathcal{S}b - b\mathcal{S}\|_{L^2\to L^2}$ follows from a theorem presented at [Jan78, p. 266], but by exploiting the relatively simple form of the Beurling transform we give a more straightforward version of the proof given in [Jan78].

When $g \in L^\infty(\mathbb{C})$ is compactly supported, the commutator of $\mathcal{S}$ and b is given by the principal value integral

(2.15)
$$\begin{aligned}(\mathcal{S}b - b\mathcal{S})g(z) &= -\frac{1}{\pi}\lim_{\epsilon\searrow 0}\int_{\mathbb{C}\setminus B(0,\epsilon)} \frac{b(w)g(w)}{(z-w)^2}\,dw + \frac{1}{\pi} b(z) \lim_{\epsilon\searrow 0}\int_{\mathbb{C}\setminus B(0,\epsilon)} \frac{g(w)}{(z-w)^2}\,dw \\ &= \frac{1}{\pi}\lim_{\epsilon\searrow 0}\int_{\mathbb{C}\setminus B(0,\epsilon)} \frac{b(z)-b(w)}{(z-w)^2} g(w)\,dw\end{aligned}$$

which converges for a.e. $z \in \mathbb{C}$.

Proposition 2.36. *If $b \in \mathrm{BMO}(\mathbb{C})$, then b satisfies*

$$\|b\|_{\mathrm{BMO}} \lesssim \|\mathcal{S}b - b\mathcal{S}\|_{L^2\to L^2}.$$

Proof. Fix a disk $B = B(z_0, r) \subset \mathbb{C}$. We intend to show that

$$\int_B |b(z) - b_B|\, dz \lesssim \|\mathcal{S}b - b\mathcal{S}\|_{L^2 \to L^2} |B|. \tag{2.16}$$

Define $s(z) := \operatorname{sgn}(b(z) - b_B)$ for every $z \in B$. Then

$$\begin{aligned}
\int_B |b(z) - b_B|\, dz &= \int_B s(z)(b(z) - b_B)\, dz = \frac{1}{|B|} \int_B s(z) \int_B (b(z) - b(w))\, dw\, dz \\
&= \frac{1}{|B|} \int_B s(z) \int_{\mathbb{C}} (z-w)^2 (b(z) - b(w)) \frac{\chi_B(w)}{(z-w)^2}\, dw\, dz.
\end{aligned}$$

By writing $(z-w)^2 = (z-z_0)^2 - 2(z-z_0)(w-z_0) + (w-z_0)^2$ for all $z, w \in \mathbb{C}$ we get

$$\begin{aligned}
\int_B |b(z) - b_B|\, dz &= \frac{\pi}{|B|} \int_B s(z)(z-z_0)^2\, (\mathcal{S}b - b\mathcal{S})\chi_B(z)\, dz \\
&- \frac{2\pi}{|B|} \int_B s(z)(z-z_0)\, (\mathcal{S}b - b\mathcal{S})((\cdot - z_0)\chi_B)(z)\, dz \\
&+ \frac{\pi}{|B|} \int_B s(z)\, (\mathcal{S}b - b\mathcal{S})((\cdot - z_0)^2 \chi_B)(z)\, dz \\
&=: I_1 + I_2 + I_3.
\end{aligned}$$

By using the fact that $|z - z_0| < r$ for every $z \in B$ and Hölder's inequality we obtain

$$\begin{aligned}
|I_1| &\leq \frac{\pi r^2}{|B|} \int_B |(\mathcal{S}b - b\mathcal{S})\chi_B(z)|\, dz \leq \|(\mathcal{S}b - b\mathcal{S})\chi_B\|_{L^2} \|\chi_B\|_{L^2} \\
&\leq \|\mathcal{S}b - b\mathcal{S}\|_{L^2 \to L^2} \|\chi_B\|_{L^2}^2 = \|\mathcal{S}b - b\mathcal{S}\|_{L^2 \to L^2} |B|,
\end{aligned}$$

and similarly $|I_2| + |I_3| \leq 3\|\mathcal{S}b - b\mathcal{S}\|_{L^2 \to L^2}|B|$. Hence, (2.16) holds. □

Theorem 2.32 follows directly from Theorem 2.27 and Propositions 2.35 and 2.36.

2.9. **Banach spaces.** We recall some notions and results of Banach space theory. Let X be a Banach space and X^* its dual. We denote the unit sphere of X by $\mathbb{S}_X$ and that of X^* by $\mathbb{S}_{X^*}$.

An element $x^* \in \mathbb{S}_{X^*}$ is said to be *norm-attaining* if there exists $x \in \mathbb{S}_X$ such that $\langle x^*, x \rangle = \|x^*\|_{X^*} \|x\|_X = 1$. It is then said that x^* attains its norm at x. The Hahn-Banach theorem implies that if $x \in \mathbb{S}_X$, then some $x^* \in X^*$ attains its norm at x (see [Con90, Corollary 6.7]).

We recall the following classical result of E. Bishop and R. Phelps (see e.g. [Meg98, Theorem 2.11.14]).

Theorem 2.37. *Norm-attaining functionals form a dense subset of* $\mathbb{S}_{X^*}$.

When we study Lagrange multipliers in Banach spaces we need the notion of Gâteaux derivative of a (possibly nonlinear) operator f between Banach spaces X and Y.

Definition 2.38. A mapping $f\colon X \to Y$ is said to be *Gâteaux differentiable* at $x \in X$ if there exists a bounded linear operator $L\colon X \to Y$ such that

$$\lim_{\mathbb{R} \ni t \to 0} \left\| \frac{f(x+th) - f(x)}{t} - Lh \right\|_X = 0$$

for every $h \in X$. The operator L is then denoted by $f'(x)$ and called the *Gâteaux derivative* of f at x.

As a special case, when $f\colon X \to \mathbb{R}$ is Gâteaux differentiable at $x \in X$, the Gâteaux derivative belongs to X^*. By a theorem of S. Banach (see [FHHMPZ01, Lemma 8.4]), the norm $\|\cdot\|_X$ is Gâteaux differentiable at $x \in \mathbb{S}_X$ if and only if there exists a *unique* element $x^* \in \mathbb{S}_{X^*}$ such that $\langle x^*, x\rangle = 1$.

Definition 2.39. When X and Y are Banach spaces and $T\colon X \to Y$ is a bounded linear operator, the kernel of T is denoted by $\ker T$ and the range of T by $\operatorname{ran} T$. The *transpose* of T is the bounded linear operator $T^*\colon Y^* \to X^*$ defined by $T^*y^* := y^* \circ T$ for every $y^* \in Y^*$.

Extreme points of convex sets are highly essential to the isometric duality theory of Banach spaces, and our setting is not an exception.

Definition 2.40. When C is a convex subset of a Banach space X, a point $x \in C$ is an *extreme point* of C if there exists no proper line segment that contains x and lies in C.

Equivalently, $x \in C$ is an extreme point of C if $x = (x_1 + x_2)/2$ for $x_1, x_2 \in C$ implies that $x_1 = x_2 = x$.

Definition 2.41. A Banach space is said to have the *Krein-Milman property* if every closed, bounded, convex set is the closed convex hull of its extreme points.

The Krein-Milman property and the related Radon-Nikodým property and dentability are discussed thoroughly, e.g., in [Bou83]. Recall the following result of C. Bessaga and A. Pełczyński (see e.g. [FHHMPZ01, Theorem 8.29]).

Theorem 2.42. *Every separable dual space has the Krein-Milman property.*

Theorem 2.42 applies, in particular, to $\mathcal{H}^1_{\mathcal{S}}(\mathbb{C})$ (see Definition 1.12 and Proposition 2.15).

Corollary 2.43. *Every closed, bounded, convex subset of $\mathcal{H}^1_{\mathcal{S}}(\mathbb{C})$ is the closed convex hull of its extreme points.*

3. The Jacobian Equation with Radial Data

Compactly supported and radial data $h \in \mathcal{H}^1(\mathbb{C})$ is one natural model case of the Jacobian equation $J_u = h$. Radial functions are easier to analyze than general datas since they have fewer degrees of freedom. They also potentially provide hints as to where to look for a counterexample to the surjectivity of $J\colon \dot{W}^{1,2}(\mathbb{C},\mathbb{C}) \to \mathcal{H}^1(\mathbb{C})$.

It is natural to look for solutions that are radial stretchings or have more general rotational symmetry properties. The goal of this chapter is to present criteria that determine whether solutions of that kind exist. As the main results we prove Theorems 1.3 and 1.4.

3.1. **Radial stretchings as solutions.** We first fix some terminology.

Definition 3.1. A measurable function $h\colon \mathbb{C} \to \mathbb{R}$ is *radial* if it is of the form $h(z) = h(|z|)$.

In many settings where the Jacobian equation is considered, if the data is radial, one expects there to exist a solution that has some radial symmetry properties, in particular a radial stretching.

Definition 3.2. A measurable mapping $u\colon \mathbb{C} \to \mathbb{C}$ is a *radial stretching* if it is of the form

$$u(z) := \rho(|z|)\frac{z}{|z|}, \tag{3.1}$$

where $\rho(r) \in \mathbb{R}$ for every $r \in [0,\infty)$.

More generally, we consider more general mappings of the form

$$u(z) := \rho(|z|)\frac{\gamma(z)}{|\gamma(z)|}, \tag{3.2}$$

where $\gamma(z) \neq 0$ a.e. Studying whether solutions of the form (3.2) exist requires careful analysis of mappings of the form (3.1). This section deals with radial stretchings, and mappings of the form (3.2) are studied in the next section.

Condition (3.1) can be made more transparent by using polar coordinates. By denoting $z = re^{i\theta}$ we may express (3.1) in the form

$$u(re^{i\theta}) = \rho(r)e^{i\theta}, \quad 0 \le r < \infty,\ 0 \le \theta < 2\pi.$$

When ρ is weakly differentiable, we denote $\dot{\rho} := d\rho/dr$. We recall a lemma about radial stretchings from [Bal82, p. 566].

Lemma 3.3. *Suppose $1 \le p < \infty$ and a measurable mapping $u\colon \mathbb{D} \to \mathbb{C}$ is of the form (3.1). Then $u \in W^{1,p}(\mathbb{D},\mathbb{C})$ if and only if ρ is absolutely continuous and*

$$\int_0^1 \left(|\dot{\rho}(r)|^p + \left|\frac{\rho(r)}{r}\right|^p \right) r\, dr < \infty. \tag{3.3}$$

Furthermore, then the Jacobian

$$J_u(z) = \frac{\dot{\rho}(|z|)\rho(|z|)}{|z|} \tag{3.4}$$

is radial.

If $u \in \dot{W}^{1,2}(\mathbb{C},\mathbb{C})$ is a radial stretching, then $\rho(0) = 0$, since otherwise the integral at (3.3) would not converge. By using (3.4) and polar coordinates we obtain

$$\int_{B(0,r)} J_u(z)\,dz = \int_0^{2\pi}\int_0^r \frac{\dot\rho(s)\rho(s)}{s}\,s\,ds\,d\theta = \pi\rho^2(r) \geq 0 \tag{3.5}$$

for every $r > 0$. If $J_u = h$, then by (3.5), ρ has to be of the form

$$\rho(r) = \pm\sqrt{\frac{1}{\pi}\int_{B(0,r)} h(z)\,dz} = \pm\sqrt{\int_0^r 2sh(s)\,ds}. \tag{3.6}$$

As (3.5) shows,

$$\int_{B(0,r)} h(z)\,dz \geq 0 \quad \text{for every } r > 0 \tag{3.7}$$

is a necessary condition for the existence of a radially symmetric solution . We next show that when combined with a strong enough regularity assumption on h, (3.7) is also a sufficient condition. We restate and prove Theorem 1.3.

Theorem 3.4. *Suppose $h \in \mathcal{H}^1(\mathbb{C})$ is compactly supported, Lipschitz continuous and radial. Then the following conditions are equivalent.*
(*i*) *There exists a radial stretching $u \in \dot{W}^{1,2}(\mathbb{C},\mathbb{C})$ that satisfies the Jacobian equation $J_u = h$.*
(*ii*) *$\int_{B(0,r)} h(z)\,dz \geq 0$ for every $r > 0$.*

Lipschitz continuity of h cannot be replaced by C^α continuity for any $\alpha \in (0,1)$ in Theorem 3.4; this is proved in Theorem 3.5.

Note that (3.5) proves direction $(i) \implies (ii)$ of Theorem 3.4. We therefore only need to prove that (ii) implies (i). For convenience we choose ρ to have a positive sign so that u is of the form

$$u(z) = \sqrt{\int_0^{|z|} 2sh(s)\,ds}\,\frac{z}{|z|}. \tag{3.8}$$

We divide the proof into several steps and first show that u satisfies the Jacobian equation.

Claim 1. *$J_u(z) = h(z)$ a.e. $z \in \mathbb{C}$.*

Proof. By (3.4) and (3.6), $J_u(z) = h(z)$ whenever $\int_0^{|z|} 2sh(s)\,ds > 0$. On the other hand, in the set where $\int_0^{|z|} 2sh(s)\,ds = 0$ we have $u(z) = 0$ a.e. and therefore $\mathrm{D}u(z) = 0$ a.e. (see [EG92, p. 130]). □

We also need to show that the mapping u defined by (3.8) belongs to $\dot{W}^{1,2}(\mathbb{C},\mathbb{C})$. Since h is compactly supported and $\int_{\mathbb{C}} h(z)\,dz = 0$, it follows that ρ is compactly supported as well, and so by Lemma 3.3 it suffices to show that ρ is Lipschitz. Choose now $R > 0$ such that

$$\mathrm{supp}(h) \subset B(0,R).$$

We show that ρ satisfies

$$\left|\frac{\rho(r_2)-\rho(r_1)}{r_2-r_1}\right| \leq \sqrt{\|h\|_{L^\infty} + 2R\|\dot h\|_{L^\infty}} \quad \text{whenever } r_1 \neq r_2$$

and divide the proof into three cases.

Claim 2. *If $r > 0$, then*

$$\left| \frac{\rho(r) - \rho(0)}{r - 0} \right| \leq \sqrt{\|h\|_{L^\infty}}.$$

Proof. We calculate

$$\left(\frac{\rho(r) - \rho(0)}{r - 0} \right)^2 = \frac{\int_0^r 2sh(s)\,ds}{r^2} \leq \|h\|_{L^\infty} \frac{\int_0^r 2s\,ds}{r^2} = \|h\|_{L^\infty}.$$

□

Claim 3. *If $r_1 \neq r_2$ are positive and $\rho(r_1) = 0$, then*

$$\left| \frac{\rho(r_2) - \rho(r_1)}{r_2 - r_1} \right| \leq \sqrt{2R\|\dot{h}\|_{L^\infty}}.$$

Proof. Suppose $r_2 > r_1$; the case $r_2 < r_1$ is proved similarly. We may assume that $r_2 \leq R$, as otherwise $\rho(r_2) = 0$.

We first show that $h(r_1) = 0$. Seeking contradiction, suppose $h(r_1) > 0$. Since

$$\int_{B(0,r_1)} h(z)\,dz = \pi\rho(r_1)^2 = 0,$$

we get $\int_{B(0,r_1-\epsilon)} h(z)\,dz < 0$ for small enough $\epsilon > 0$, contradicting (ii). Similarly, $h(r_1) < 0$ would lead to a contradiction, and so $h(r_1) = 0$.

Since $\rho(r_1)^2 = \int_0^{r_1} 2rh(r)\,dr = 0$, we may write $\rho(r_2)^2 = \int_{r_1}^{r_2} 2rh(r)\,dr$, and so, using the fact that $h(r_1) = 0$,

$$\begin{aligned} \left(\frac{\rho(r_2) - \rho(r_1)}{r_2 - r_1} \right)^2 &= \frac{\int_{r_1}^{r_2} 2rh(r)\,dr}{(r_2 - r_1)^2} \\ &= \fint_{r_1}^{r_2} 2r \frac{h(r) - h(r_1)}{r - r_1} \frac{r - r_1}{r_2 - r_1}\,dr \\ &\leq \|\dot{h}\|_{L^\infty} \fint_{r_1}^{r_2} 2r \leq 2R\|\dot{h}\|_{L^\infty}. \end{aligned}$$

□

Note that if $\rho(r) = \sqrt{\int_0^r 2sh(s)\,ds} > 0$, then ρ is continuously differentiable at r and

$$\dot{\rho}(r) = \frac{rh(r)}{\rho(r)}.$$

Recall that $\mathrm{supp}(h) \subset B(0, R)$.

Claim 4. *If $0 < r_1 < r_2$ and $\rho(r_1), \rho(r_2) > 0$, then*

$$\left| \frac{\rho(r_2) - \rho(r_1)}{r_2 - r_1} \right| \leq \sqrt{\|h\|_{L^\infty} + 2R\|\dot{h}\|_{L^\infty}}.$$

Proof. First suppose there exists $r \in (r_1, r_2)$ such that $\rho(r) = 0$. Suppose $\rho(r_2) > \rho(r_1)$; the case $\rho(r_2) \leq \rho(r_1)$ is treated analogously. Then the trivial estimate $|\rho(r_2) - \rho(r_1)| < |\rho(r_2) - \rho(r)|$ and Claim 3 yield

$$\left| \frac{\rho(r_2) - \rho(r_1)}{r_2 - r_1} \right| < \left| \frac{\rho(r_2) - \rho(r)}{r_2 - r} \right| \leq \sqrt{2R\|\dot{h}\|_{L^\infty}}.$$

Suppose next $\rho > 0$ in the interval $[r_1, r_2]$. Use the Mean value theorem to write

$$\frac{\rho(r_2) - \rho(r_1)}{r_2 - r_1} = \dot{\rho}(r)$$

for some $r \in (r_1, r_2)$. It thus suffices to show that $|\dot{\rho}(r)| \leq \sqrt{\|h\|_{L^\infty} + 2R\|\dot{h}\|_{L^\infty}}$ whenever $r_1 \leq r \leq r_2$.

We initially have little control over the values $\dot{\rho}(r_1)$ and $\dot{\rho}(r_2)$ and we therefore want to consider a larger interval where $\dot{\rho}$ satisfies useful bounds at the endpoints. Let

$$r_3 := \max\{r < r_1 \colon \rho(r) = 0\} \quad \text{and} \quad r_4 := \min\{r > r_2 \colon \rho(r) = 0\};$$

note that r_3 and r_4 exist since $\rho(0) = 0$ and ρ is compactly supported. Since $\rho(r_3) = 0$, Claim 3 implies that the average of $\dot{\rho}$ over $[r_3, r_1]$ satisfies

$$\left| \fint_{r_3}^{r_1} \dot{\rho}(r)\, dr \right| = \left| \frac{\rho(r_1) - \rho(r_3)}{r_1 - r_3} \right| \leq \sqrt{2R\|\dot{h}\|_{L^\infty}}.$$

We may thus choose $\tilde{r_1}$ such that

$$r_3 < \tilde{r_1} < r_1 \quad \text{and} \quad |\dot{\rho}(\tilde{r_1})| \leq \sqrt{2R\|\dot{h}\|_{L^\infty}}. \tag{3.9}$$

Similarly, we choose $\tilde{r_2}$ such that

$$r_2 < \tilde{r_2} < r_4 \quad \text{and} \quad |\dot{\rho}(\tilde{r_2})| \leq \sqrt{2R\|\dot{h}\|_{L^\infty}}. \tag{3.10}$$

Now $[r_1, r_2] \subset [\tilde{r_1}, \tilde{r_2}]$ and so it suffices to show that

$$|\dot{\rho}(r)| \leq \sqrt{\|h\|_{L^\infty} + 2R\|\dot{h}\|_{L^\infty}} \quad \text{for every } r \in [\tilde{r_1}, \tilde{r_2}]. \tag{3.11}$$

We now smoothen h and obtain (3.11) by approximation. To that end, fix a mollifier $\Psi \in C_0^\infty(\mathbb{R})$ such that $\operatorname{supp}(\Psi) \subset [-1, 1]$ and $\int_{-1}^1 \Psi(r)\, dr = 1$. When $t > 0$, denote $\Psi_t(r) := t^{-1}\Psi(r/t)$ so that $\int_{-\infty}^\infty \Psi_t(r)\, dr = 1$. Define $h_t \colon \mathbb{R} \to \mathbb{R}$ as the one-dimensional convolution

$$h_t(r) := \int_{-\infty}^\infty h(s)\psi_t(r - s)\, ds.$$

Clearly $\|h_t\|_{L^\infty} \leq \|h\|_{L^\infty}$ and $\|\dot{h}_t\|_{L^\infty} \leq \|\dot{h}\|_{L^\infty}$. Furthermore, $\|h_t - h\|_{L^\infty} \to 0$ as $t \searrow 0$ since h is Lipschitz continuous.

Since $\rho(r)^2 = \int_0^r 2sh(s)\, ds$ is bounded away from zero in $[\tilde{r_1}, \tilde{r_2}]$, for small enough $t > 0$ we may define

$$\rho_t(r) := \sqrt{\int_0^r 2sh_t(s)\, ds} > 0 \quad \text{for } r \in [\tilde{r_1}, \tilde{r_2}].$$

Then

$$\dot{\rho}_t(r) = \frac{rh_t(r)}{\rho_t(r)} \to \dot{\rho}(r) \quad \text{uniformly in } [\tilde{r_1}, \tilde{r_2}] \tag{3.12}$$

when $t \searrow 0$. We will use (3.12) to prove (3.11).

The smooth function $\dot{\rho}_t$ attains its maximum in $[\tilde{r_1}, \tilde{r_2}]$ either at one of the endpoints $\tilde{r_1}$ and $\tilde{r_2}$ or at a point where the second derivative $\ddot{\rho}_t$ vanishes. In the endpoints (3.9), (3.10) and (3.12) imply that

$$\max\{|\dot{\rho}_t(\tilde{r_1})|, |\dot{\rho}_t(\tilde{r_2})|\} \le \sqrt{2R\|\dot{h}\|_{L^\infty}} + o(t)$$

as $t \searrow 0$. Furthermore, if

$$\ddot{\rho}_t(r) = \frac{h_t(r) + r\dot{h}_t(r) - \frac{r^2 h_t(r)^2}{\rho_t(r)^2}}{\rho_t(r)} = 0,$$

then

$$\dot{\rho}_t(r)^2 = \frac{r^2 h_t(r)^2}{\rho_t(r)^2} = h_t(r) + r\dot{h}_t(r) \le \|h_t\|_{L^\infty} + R\|\dot{h}_t\|_{L^\infty} \le \|h\|_{L^\infty} + R\|\dot{h}\|_{L^\infty}.$$

We conclude that

$$\max_{r\in[\tilde{r_1},\tilde{r_2}]} |\dot{\rho}_t(r)| \le \sqrt{\|h\|_{L^\infty} + 2R\|\dot{h}\|_{L^\infty}} + o(t) \quad \text{as } t \searrow 0.$$

Combining this estimate with (3.12) yields (3.11), and the claim follows.

We have now covered all the cases. We thus conclude that ρ is Lipschitz continuous and Theorem 3.4 holds. □

Lipschitz continuity cannot be replaced in Theorem 3.4 by α-Hölder continuity for any $0 < \alpha < 1$, as we show in the following result.

Theorem 3.5. *Suppose* $0 < \alpha < 1$. *Then there exists a compactly supported, radially symmetric function* $h \in \mathcal{H}^1(\mathbb{C}) \cap C^\alpha(\mathbb{C})$ *such that* $\int_{B(0,r)} h(z)\,dz \ge 0$ *for every* $r > 0$ *but there exists no radial stretching* $u \in \dot{W}^{1,2}(\mathbb{C},\mathbb{C})$ *such that* $J_u = h$.

Proof. In order to construct a suitable function h set $a_1 = 1$ and

$$a_{k+1} = a_k + \frac{4}{k^{\frac{1}{\alpha}}} = 1 + \sum_{j=1}^{k} \frac{4}{j^{\frac{1}{\alpha}}}$$

for every $k \in \mathbb{N}$. Given $k \in \mathbb{N}$ define

$$h(r) := \begin{cases} \frac{(r-a_k)^\alpha}{r}, & a_k \le r < a_k + \frac{1}{k^{\frac{1}{\alpha}}}, \\ \frac{(a_k+2k^{-\frac{1}{\alpha}}-r)^\alpha}{r}, & a_k + \frac{1}{k^{\frac{1}{\alpha}}} \le r < a_k + \frac{2}{k^{\frac{1}{\alpha}}}, \\ \frac{-(r-a_k-2k^{-\frac{1}{\alpha}})^\alpha}{r}, & a_k + \frac{2}{k^{\frac{1}{\alpha}}} \le r < a_k + \frac{3}{k^{\frac{1}{\alpha}}}, \\ \frac{-(a_{k+1}-r)^\alpha}{r}, & a_k + \frac{3}{k^{\frac{1}{\alpha}}} \le r \le a_{k+1} \end{cases}$$

so that $\int_{a_k}^{a_{k+1}} 2rh(r)\,dr = 0$ as the integrals over the four subintervals of $[a_k, a_{k+1}]$ cancel out. Set $h(r) := 0$ when

$$r \notin \left[1, 1 + \sum_{j=1}^{\infty} \frac{4}{j^{\frac{1}{\alpha}}}\right].$$

Now h has all the required properties.

We next show that a mapping of the form $u(z) = \rho(|z|)z/|z|$, where $\rho(|z|) = \pm\sqrt{\int_0^{|z|} 2rh(r)\,dr}$ a.e. $z \in \mathbb{C}$, cannot belong to $\dot{W}^{1,2}(\mathbb{C},\mathbb{C})$. By Lemma 3.3, it suffices to show that

$$\int_0^\infty r\dot{\rho}(r)^2 dr = \infty.$$

Fix $k \in \mathbb{N}$ and use the condition $\int_0^{a_k} 2sh(s)\,ds = 0$ to calculate

$$\begin{aligned}
\int_{a_k}^{a_k+k^{-\frac{1}{\alpha}}} r\dot{\rho}(r)^2 dr &= \int_{a_k}^{a_k+k^{-\frac{1}{\alpha}}} \frac{r^3h(r)^2}{\int_0^r 2sh(s)\,ds}\,dr \\
&= \int_{a_k}^{a_k+k^{-\frac{1}{\alpha}}} \frac{(r-a_k)^{2\alpha}r}{\int_{a_k}^r 2(s-a_k)^\alpha\,ds}\,dr \\
&\gtrsim_\alpha \int_{a_k}^{a_k+k^{-\frac{1}{\alpha}}} (r-a_k)^{\alpha-1}dr \gtrsim_\alpha \frac{1}{k}.
\end{aligned}$$

Thus

$$\int_0^\infty r\dot{\rho}(r)^2 dr \gtrsim_\alpha \sum_{k=1}^\infty \frac{1}{k},$$

and since the series $\sum_{k=1}^\infty k^{-1}$ diverges, we conclude that $u \notin \dot{W}^{1,2}(\mathbb{C},\mathbb{C})$. $\square$

3.2. **Generalized radially symmetric solutions.** As formula (3.5) shows, if $u \in \dot{W}^{1,2}(\mathbb{C},\mathbb{C})$ is a radial stretching, then

$$\int_{B(0,r)} J_u(z)\,dz \geq 0 \quad \text{for every } r > 0,$$

and this puts a severe restriction on solving the Jacobian equation. However, the complex conjugate $\bar{u}$ is then of the form

$$\bar{u}(z) = \rho(|z|)\frac{\bar{z}}{|z|},$$

and since $J_{\bar{u}} = -J_u$, we have $\int_{B(0,r)} J_{\bar{u}}(z)\,dz \leq 0$ for all $r > 0$. We would like to allow the integrals $\int_{B(0,r)} J_u(z)\,dz$ to change sign as r varies, and in particular, we wish to allow u to be of the form $u(z) = \rho(|z|)z/|z|$ for some $|z|$ and $u(z) = \rho(|z|)\bar{z}/|z|$ for others.

We generalize radial stretchings by only assuming that $\rho = |u|$ is radially symmetric; u is then of the form

$$u(z) = \rho(|z|)\frac{\gamma(z)}{|\gamma(z)|}, \tag{3.13}$$

where $\gamma(z) \neq 0$ a.e. Radial stretchings are a special case where $\gamma(z) = z$. In polar coordinate notation a radial mapping is of the form $u(re^{i\theta}) = \rho(r)e^{i\theta}$ whereas (3.13) can be written (formally) as $u(re^{i\theta}) = \rho(r)e^{i\alpha(re^{i\theta})}$.

Even this generalization of radial stretchings does not allow us to solve the Jacobian equation with general radial data, as the following theorem shows.

Theorem 3.6. *Suppose $h \in \mathcal{H}^1(\mathbb{C})$ is compactly supported, Lipschitz continuous and radial. Then the following conditions are equivalent.*

(i) There exists a mapping $u \in \dot{W}^{1,2}(\mathbb{C},\mathbb{C})$ of the form (3.13) that satisfies

the Jacobian equation $J_u = h$.
(ii) If $r > 0$ and $\int_{B(0,r)} h(z)\,dz = 0$, then $h(r) = 0$.

Before beginning the proof we set the following notation.

Definition 3.7. When $0 \le a < b < \infty$, the open annulus determined by a and b is denoted by

$$\mathbb{A}(a,b) := \{z \in \mathbb{C} \colon a < |z| < b\}.$$

The closure of $\mathbb{A}(a,b)$ is denoted by $\bar{\mathbb{A}}(a,b)$.

We first prove direction $(i) \implies (ii)$. Seeking contradiction, assume $\int_{B(0,r_0)} h(z)\,dz = 0$ and $h(r_0) > 0$ for some $r_0 > 0$ and that u is of the form (3.13) and solves the Jacobian equation. We divide the proof by contradiction into three steps.

Claim 1. *There exists $\delta > 0$ such that in the annulus $\mathbb{A}(r_0 - 2\delta, r_0 + 2\delta)$ the mapping u is continuous and $h, \rho \ge C > 0$.*

Proof. Since h is continuous and $h(r_0) > 0$, we can choose $\delta > 0$ such that $h(r) \ge C$ when $r_0 - 2\delta \le r \le r_0 + 2\delta$. Hence, u has integrable distortion $K_u = (|u_z| + |u_{\bar z}|)^2 / J_u \le 2|\mathrm{D}u|^2/C$ in the annulus $\mathbb{A}(r_0 - 2\delta, r_0 + 2\delta)$. By Theorem 2.7, u is continuous in $\mathbb{A}(r_0 - 2\delta, r_0 + 2\delta)$. Thus ρ is continuous in $\mathbb{A}(r_0 - 2\delta, r_0 + 2\delta)$, and so by possibly choosing a smaller annulus it suffices to show that $\rho(r_0) > 0$.

Suppose, by way of contradiction, that $\rho(r_0) = 0$. By continuity, $u(z) = 0$ for every $z \in \partial B(0, r_0)$. As a consequence,

$$v := u\chi_{B(0,r_0)} \in \dot W^{1,2}(\mathbb{C}, \mathbb{C}).$$

The linear distortion of v is given by $K_v = K_u$ when $r_0 - 2\delta < |z| < r_0$ and $K_v(z) = 1$ when $r_0 < |z| < r_0 + 2\delta$. Thus K_v is integrable in the annulus $\mathbb{A}(r_0 - 2\delta, r_0 + 2\delta)$, and by Theorem 2.7, v is either constant or an open mapping there. However, v is not constant in $\mathbb{A}(r_0 - 2\delta, r_0)$ (since $J_v = h \ge C$ in $\mathbb{A}(r_0 - 2\delta, r_0)$) and not open in $\mathbb{A}(r_0, r_0 + 2\delta)$. We have reached a contradiction, and therefore $\rho(r_0) > 0$. $\square$

The rest of the proof uses a winding number argument, and in order to ensure that the calculations remain rigorous we approximate u. Recall that we fixed in §2.3 a function $\Phi \in C_0^\infty(\mathbb{C})$ that satisfies $\mathrm{supp}(\Phi) \subset B$ and $\Phi_t(z) := \Phi(z/t)/t^2$ when $t > 0$ and $z \in \mathbb{C}$. Consider smooth mappings of the form

$$u_t(z) := \rho(|z|) \frac{u * \Phi_t(z)}{|u * \Phi_t(z)|}, \quad 0 < t < \delta,$$

in $\mathbb{A}(r_0 - \delta, r_0 + \delta)$. We use the chain rule to write

$$\mathrm{D}\frac{u * \Phi_t}{|u * \Phi_t|} = \begin{bmatrix} \frac{y^2}{|z|^3} & \frac{-xy}{|z|^3} \\ \frac{-xy}{|z|^3} & \frac{x^2}{|z|^3} \end{bmatrix} \circ (u * \Phi_t)\, \mathrm{D}(u * \Phi_t)$$

By Claim 1, u and $\rho = |u|$ are continuous in $\mathbb{A}(r_0 - 2\delta, r_0 + 2\delta)$, and therefore the convergence $u * \Phi_t \to u$ is uniform in $\mathbb{A}(r_0 - \delta, r_0 + \delta)$. In particular, by Claim 1, $|u * \Phi_t| \ge C' > 0$ for small enough $t > 0$. Consequently, $\|\mathrm{D}u_t - \mathrm{D}u\|_{L^2(\mathbb{A}(r_0 - \delta, r_0 + \delta))} \to 0$ as $t \searrow 0$. Therefore,

$$\int_{\mathbb{A}(r_0 - \delta, r_0 + \delta)} J_{u_t}(z)\,dz \to \int_{\mathbb{A}(r_0 - \delta, r_0 + \delta)} J_u(z)\,dz \quad \text{as } t \searrow 0. \tag{3.14}$$

We use Claim 1 to choose $t > 0$ small enough that $u * \Phi_t$ is bounded away from zero when $r_0 - \delta < |z| < r_0 + \delta$. We can thus write, for $t > 0$ small and $z \in \mathbb{A}(r_0 - \delta, r_0)$,

$$\frac{u * \Phi_t(z)}{|u * \Phi_t(z)|} = e^{i\alpha_t(z)},$$

where, written in terms of polar coordinates, the mapping

$$(r, \theta) \mapsto \alpha_t(re^{i\theta})\colon (r_0 - \delta, r_0) \times (0, 2\pi) \to \mathbb{R}$$

is smooth.

Since u_t is continuous and non-zero, the winding number of the closed path $\theta \mapsto e^{i\alpha_t(re^{i\theta})}\colon [0, 2\pi] \to \mathbb{C}$ around the origin is constant with respect to r in the interval $(r_0 - \delta, r_0)$, and we denote

$$W_t := \frac{1}{2\pi} \int_0^{2\pi} \partial_\theta \alpha_t(re^{i\theta})\, d\theta \quad \text{for every } r \in (r_0 - \delta, r_0). \tag{3.15}$$

In Claims 2 and 3 we find the contradiction we are seeking by proving that $W_t \in \mathbb{Z} \setminus \{0\}$ for small $t > 0$ but $\lim_{t \searrow 0} W_t = 0$.

Claim 2. $W_t \in \mathbb{Z} \setminus \{0\}$ *for all small enough* $t > 0$.

Proof. Claim 1 implies that $\int_{\mathbb{A}(r_0-\delta,r_0)} J_u(z)\, dz > 0$, and by (3.14) there exists $t_0 > 0$ such that when $0 < t < t_0$, we have $\int_{\mathbb{A}(r_0-\delta,r_0)} J_{u_t}(z)\, dz > 0$. Let $0 < t < t_0$. When $r_0 - \delta < r < r_0$ and $\theta \in (0, 2\pi)$, we write $u_t(z) = \rho(|z|)(\cos \alpha_t(z) + i \sin \alpha_t(z))$ and calculate

$$J_{u_t}(z) = \frac{\dot\rho(|z|)\rho(|z|)}{|z|}(x\partial_y \alpha_t(z) - y\partial_x \alpha_t(z)) = \frac{\dot\rho(|z|)\rho(|z|)}{|z|}\, \partial_\theta \alpha_t(z). \tag{3.16}$$

By combining this with (3.15) we get

$$0 < \int_{\mathbb{A}(r_0-\delta,r_0)} J_{u_t}(z)\, dz = 2\pi W_t \int_{r_0-\delta}^{r_0} \dot\rho(r)\rho(r)\, dr.$$

As a consequence, $W_t \neq 0$. □

Claim 3. $\lim_{t \searrow 0} W_t = 0$.

Proof. Choose a smooth radial cutoff function $\psi\colon \mathbb{C} \to [0, 1]$ that satisfies

$$\psi|_{[r_0-\delta/2, r_0+\delta/2]} = 1 \quad \text{and} \quad \psi|_{[0, r_0-\delta] \cup [r_0+\delta, \infty)} = 0. \tag{3.17}$$

Since the Jacobian is a null Lagrangian (see Proposition 2.21) and $\|u_t - u\|_{W^{1,2}(\mathbb{A}(r_0-\delta,r_0))} \to 0$ when $t \searrow 0$, we obtain

$$0 = \int_{B(0,r_0)} J_u(z)\, dz = \int_{\mathbb{A}(r_0-\delta,r_0)} J_{\psi u}(z)\, dz = \lim_{t \searrow 0} \int_{\mathbb{A}(r_0-\delta,r_0)} J_{\psi u_t}(z)\, dz. \tag{3.18}$$

When $r_0 - \delta < |z| = r < r_0$, we write, as in (3.16),

$$J_{\psi u_t} = \frac{\partial_r(\psi\rho)\psi\rho}{|z|}\, \partial_\theta \alpha_t.$$

Thus, using (3.15) and (3.17),

$$\begin{aligned} \int_{\mathbb{A}(r_0-\delta,r_0)} J_{\psi u_t}(z)\, dz &= \int_{r_0-\delta}^{r_0} \partial_r(\psi\rho)(r)\psi(r)\rho(r) \int_0^{2\pi} \partial_\theta \alpha_t(re^{i\theta})\, d\theta\, dr \\ &= \pi\rho(r_0)^2 W_t. \end{aligned}$$

By combining this equality with (3.18) we obtain

$$\pi\rho(r_0)^2 \lim_{t\searrow 0} W_t = \int_{B(0,r_0)} h(z)\,dz = 0,$$

and since $\rho(r_0) > 0$ by Claim 1, the proof is complete. □

We have reached a contradiction by proving Claims 2 and 3, and so the proof of direction $(i) \implies (ii)$ of Theorem 3.6 is complete. We now prove the converse.

Proof of direction $(ii) \implies (i)$. First express h in the form

$$h = h\chi_{\int_0^{|z|} 2sh(s)\,ds>0} + h\chi_{\int_0^{|z|} 2sh(s)\,ds<0} =: h_1 + h_2,$$

where $\chi_{\int_0^{|z|} 2sh(s)\,ds>0}$ and $\chi_{\int_0^{|z|} 2sh(s)\,ds<0}$ are characteristic functions. By assumption (ii), $h(z) = 0$ whenever $\int_0^{|z|} 2sh(s)\,ds = 0$, and it easily follows that h_1 and h_2 are Lipschitz continuous.

We express the open set $\{r \in (0,\infty)\colon \int_0^r 2sh(s)\,ds > 0\}$ as an at most countably infinite union of disjoint intervals $\cup_{j\in J}(a_j, b_j)$. Note that $\int_0^{a_j} rh(r)\,dr = \int_0^{b_j} rh(r)\,dr = 0$ for every $j \in J$ and calculate

$$\int_{\mathbb{C}} h_1(z)\,dz = 2\pi \int_0^\infty rh(r) \sum_{j\in J} \chi_{(a_j,b_j)}(r)\,dr = 2\pi \sum_{j\in J} \int_{a_j}^{b_j} rh(r)\,dr = 0.$$

Thus $h_1 \in \mathcal{H}^1(\mathbb{C})$, and clearly $\int_{B(0,r)} h_1(z)\,dz \geq 0$ for every $r > 0$. Similarly, $h_2 \in \mathcal{H}^1(\mathbb{C})$ satisfies $\int_{B(0,r)} h_2(z)\,dz \leq 0$ for all $r > 0$. We can thus apply Theorem 3.4 to h_1 and h_2.

Define $u\colon \mathbb{C} \to \mathbb{C}$ by

$$\begin{aligned} & u(z) \\ := \; & \chi_{\int_0^{|z|} 2sh_1(s)\,ds>0}(z)\sqrt{\int_0^{|z|} 2sh_1(s)\,ds}\,\frac{z}{|z|} \\ + \; & \chi_{\int_0^{|z|} 2sh_2(s)\,ds<0}(z)\sqrt{-\int_0^{|z|} 2sh_2(s)\,ds}\,\frac{\bar{z}}{|z|}. \end{aligned}$$

When $\int_0^{|z|} 2sh(s)\,ds > 0$, choose $j \in J$ such that $|z| \in (a_j, b_j)$ and calculate

$$\int_0^{|z|} 2sh_1(z)\,ds = \int_{a_j}^{|z|} 2sh_1(s)\,ds = \int_{a_j}^{|z|} 2sh(s)\,ds = \int_0^{|z|} 2sh(s)\,ds.$$

Similarly, $\int_0^{|z|} 2sh_2(z)\,ds = \int_0^{|z|} 2sh(s)\,ds$ when $\int_0^{|z|} 2sh(s)\,ds < 0$.

Theorem 3.4 implies that $u \in \dot{W}^{1,2}(\mathbb{C},\mathbb{C})$ and that $J_u = h_1 = h$ when $\int_0^{|z|} 2sh(s)\,ds \geq 0$ and $J_u = h_2 = h$ when $\int_0^{|z|} 2sh(s)\,ds \leq 0$. □

If a compactly supported, Lipschitz continuous function $h \in \mathcal{H}^1(\mathbb{C})$ satisfies $h(r) > 0$ and $\int_{B(0,r)} h = 0$ for some $r > 0$, Theorem 3.6 says that the Jacobian equation $J_u = h$ cannot have a solution of the generalized radially symmetric form (3.13). However, at least in some cases there exists a solution without such radial symmetry properties; this follows from [CDK09, Theorem 1]. To the author's knowledge, it remains an open problem whether the Jacobian equation always has a solution when the data is radial.

4. Energy-Minimal Solutions and Lagrange Multipliers

While in §3 we studied the Jacobian equation in a concrete model case, in this chapter and the two following ones we use abstract functional analytic techniques to study the Jacobian equation in more generality. We recall the definition of energy-minimal solutions from the Introduction.

Definition 4.1. If a mapping $u \in \dot{W}^{1,2}(\mathbb{C},\mathbb{C})$ satisfies

$$\int_{\mathbb{C}} |u_{\bar{z}}|^2 = \min\left\{\int_{\mathbb{C}} |v_{\bar{z}}|^2 \colon v \in \dot{W}^{1,2}(\mathbb{C},\mathbb{C}),\ J_v = J_u\right\},$$

then we call u an *energy minimizer* or an *energy-minimal solution.*

The study of energy-minimal solutions begins in §4.1. The range of $J \colon \dot{W}^{1,2}(\mathbb{C},\mathbb{C}) \to \mathcal{H}^1(\mathbb{C})$ is dense in $\mathcal{H}^1(\mathbb{C})$ (see Corollary 4.4). Thus, if all energy-minimal solutions satisfy the uniform estimate

$$\int_{\mathbb{C}} |u_{\bar{z}}|^2 \lesssim \|J_u\|_{\mathcal{H}^1}, \tag{4.1}$$

then J maps $\dot{W}^{1,2}(\mathbb{C},\mathbb{C})$ onto $\mathcal{H}^1(\mathbb{C})$ (see Proposition 4.5). In order to construct a large class of minimizers that satisfy (4.1) we use Lagrange multipliers whose definition we give in §4.2. Chapters 5 and 6 deal with Lagrange multipliers more extensively. The main goal is to prove Theorems 1.26 and 1.27.

4.1. **Energy-minimal solutions.** If the Jacobian equation has a solution, then it has an energy-minimal solution. We prove this standard fact by using the direct method of the calculus of variations.

Proposition 4.2. *Let $h \in \mathcal{H}^1(\mathbb{C})$ and suppose there exists a solution $v \in \dot{W}^{1,2}(\mathbb{C},\mathbb{C})$ of the Jacobian equation $J_v = h$. Then there exists an energy-minimal solution $u \in \dot{W}^{1,2}(\mathbb{C},\mathbb{C})$ with $J_u = h$, that is,*

$$\int_{\mathbb{C}} |u_{\bar{z}}|^2 = \min\left\{\int_{\mathbb{C}} |v_{\bar{z}}|^2 \colon J_v = h\right\}. \tag{4.2}$$

Proof. Choose a minimizing sequence so that $J_{u^j} = h$ for every $j \in \mathbb{N}$ and

$$\lim_{j\to\infty} \int_{\mathbb{C}} |u^j_{\bar{z}}|^2 = \inf\left\{\int_{\mathbb{C}} |v_{\bar{z}}|^2 \colon J_v = h\right\}.$$

The sequence of the values $\int_{\mathbb{C}} |u^j_{\bar{z}}|^2$ is bounded. Since the unit ball of $L^2(\mathbb{C},\mathbb{C})$ is weakly compact, there exists $g \in L^2(\mathbb{C},\mathbb{C})$ such that, for a subsequence which we do not relabel,

$$u^j_{\bar{z}} \rightharpoonup g \tag{4.3}$$

in $L^2(\mathbb{C},\mathbb{C})$.

By Lemma 2.26 there exists $u \in \dot{W}^{1,2}(\mathbb{C},\mathbb{C})$ such that $u_{\bar{z}} = g$. By (4.3) and Proposition 2.30, $h = J_{u^j} = |\mathcal{S}u^j_{\bar{z}}|^2 - |u^j_{\bar{z}}|^2 \overset{*}{\rightharpoonup} |\mathcal{S}g|^2 - |g|^2 = J_u$, and so $J_u = h$. Since the L^2 norm is weakly lower semicontinuous, it follows that

$$\int_{\mathbb{C}} |u_{\bar{z}}|^2 \le \liminf_{j\to\infty} \int_{\mathbb{C}} |u^j_{\bar{z}}|^2.$$

The mapping u thus satisfies (4.2). □

For the approach to the Jacobian equation via energy-minimal solutions to make sense, it is crucial that the range of the Jacobian operator is dense in $\mathcal{H}^1(\mathbb{C})$. Extending techniques introduced in [CDK09] O. Kneuss proved in [Kne12] the following deep result.

Theorem 4.3. *Suppose $h \in C_0^1(\mathbb{D})$ satisfies $\int_{\mathbb{D}} h = 0$. Then*

$$\inf_{u \in C_0^\infty(\mathbb{D},\mathbb{C})} \|J_u - h\|_{C^1(\mathbb{D})} = 0.$$

Recall that by Proposition 2.14, $C_\bullet^\infty(\mathbb{C})$ is dense in $\mathcal{H}^1(\mathbb{C})$. Proposition 2.11 and Theorem 4.3 then imply the following important result.

Corollary 4.4. *The range of the operator $J \colon \dot{W}^{1,2}(\mathbb{C},\mathbb{C}) \to \mathcal{H}^1(\mathbb{C})$ is dense in $\mathcal{H}^1(\mathbb{C})$.*

Corollary 4.4 can also be deduced from the earlier result [CDK09, Theorem 1] with a little bit of work.

Since the range of J is dense in $\mathcal{H}^1(\mathbb{C})$, proving the *a priori* estimate

$$\int_{\mathbb{C}} |u_{\bar{z}}|^2 \lesssim \|J_u\|_{\mathcal{H}^1} \tag{4.4}$$

for energy-minimal solutions would show that $J \colon \dot{W}^{1,2}(\mathbb{C},\mathbb{C}) \to \mathcal{H}^1(\mathbb{C})$ is surjective. We prove this fact for completeness.

Proposition 4.5. *Suppose every energy-minimal solution satisfies (4.4). Then J maps $\dot{W}^{1,2}(\mathbb{C},\mathbb{C})$ onto $\mathcal{H}^1(\mathbb{C})$.*

Proof. Let $h \in \mathcal{H}^1(\mathbb{C})$. By using Corollary 4.4 select a sequence $(v^j)_{j=1}^\infty$ in $\dot{W}^{1,2}(\mathbb{C},\mathbb{C})$ such that $\lim_{j\to\infty} \|J_{v^j} - h\|_{\mathcal{H}^1} = 0$. Choose energy-minimal solutions $u^j \in \dot{W}^{1,2}(\mathbb{C},\mathbb{C})$ that satisfy $J_{u^j} = J_{v^j}$ and

$$\int_{\mathbb{C}} |u^j_{\bar{z}}|^2 \lesssim \|J_{u^j}\|_{\mathcal{H}^1} \leq \|h\|_{\mathcal{H}^1} + 1$$

from some index on.

Again, for a subsequence, $u^j_{\bar{z}} \rightharpoonup g$ in $L^2(\mathbb{C},\mathbb{C})$, and by Lemma 2.26 there exists $u \in \dot{W}^{1,2}(\mathbb{C},\mathbb{C})$ such that $u_{\bar{z}} = g$. By Proposition 2.30, $J_{u^j} \overset{*}{\rightharpoonup} J_u$ in $\mathcal{H}^1(\mathbb{C})$, and so $J_u = h$. □

Remark 4.6. By focusing on energy-minimal solutions we also gain more control on the behavior of the solutions considered. As an example of this, if $u \in \dot{W}^{1,2}(\mathbb{C},\mathbb{C})$ is an energy-minimal solution and $J_u = 0$ a.e. in a domain $\Omega \subset \mathbb{C}$, then $u \in W^{1,\infty}_{loc}(\Omega,\mathbb{C})$ and $|u(\Omega)| = 0$ (see Proposition 8.4).

If the minimality assumption is dropped, the mapping may exhibit rather wild behavior. There exists, for instance, a continuous mapping $u \in W^{1,2}(\mathbb{C},\mathbb{C})$ such that $J_u = 0$ a.e. in $\mathbb{C}$ and u maps the line segment $[0,1] \times \{0\}$ onto the square $[0,1]^2$ (see [MM95, p. 34]). Furthermore, the assumption that $J_u = 0$ in a domain $\Omega \subset \mathbb{C}$ does not, in general, imply additional integrability for $\mathrm{D}u$. This is seen easily by choosing any $u_1 \in \dot{W}^{1,2}(\mathbb{C})$ and setting $u_2 = 0$ in Ω.

Our main point of focus is the question whether every energy-minimal solution satisfies (4.4). The author has been unable to solve this problem but a multitude of partial results are proved in this dissertation.

4.2. **Lagrange multipliers.** In multidimensional calculus Lagrange multipliers are used to minimize a function subject to a constraint. We use a similar strategy in Banach spaces; in our case the minimization problem is

$$\text{minimize} \int_{\mathbb{C}} |u_{\bar z}|^2 \text{ subject to } J_u = h,$$

where $h \in \mathcal{H}^1(\mathbb{C})$ is given. Lagrange multipliers in Banach spaces are treated, for instance, in chapter 43 of [Zei85]. In order to get started we introduce some notations and definitions.

When $u, \phi \in \dot W^{1,2}(\mathbb{C}, \mathbb{C})$, the Gâteaux derivative of the functional $\|\cdot\|^2_{\dot W^{1,2}}$ at u in the direction ϕ is

$$\frac{d}{d\epsilon} \int_{\mathbb{C}} |(u+\epsilon\phi)_{\bar z}|^2 \bigg|_{\epsilon=0} = 2\mathbf{Re} \int_{\mathbb{C}} u_{\bar z}\overline{\phi_{\bar z}}.$$

Similarly, the Gâteaux derivative of the Jacobian operator J is of the form

$$J'_u\phi := \frac{d}{d\epsilon} J_{u+\epsilon\phi}\bigg|_{\epsilon=0} = J_{u_1+i\phi_2} + J_{\phi_1+iu_2} = 2\mathbf{Re}(u_z\overline{\phi_z} - u_{\bar z}\overline{\phi_{\bar z}}). \tag{4.5}$$

In particular, $J'_u u = 2J_u$.

Definition 4.7. A function $b \in \mathrm{BMO}(\mathbb{C})$ is called a *Lagrange multiplier* for $u \in \dot W^{1,2}(\mathbb{C}, \mathbb{C})$ if

$$\int^*_{\mathbb{C}} b \,\frac{d}{d\epsilon} J_{u+\epsilon\phi}\bigg|_{\epsilon=0} = \frac{d}{d\epsilon}\int_{\mathbb{C}} |(u+\epsilon\phi)_{\bar z}|^2\bigg|_{\epsilon=0}$$

for every $\phi \in \dot W^{1,2}(\mathbb{C}, \mathbb{C})$, that is,

$$\int^*_{\mathbb{C}} b\mathbf{Re}(u_z\overline{\phi_z} - u_{\bar z}\overline{\phi_{\bar z}}) = \mathbf{Re}\int_{\mathbb{C}} u_{\bar z}\overline{\phi_{\bar z}} \quad \text{for all } \phi \in \dot W^{1,2}(\mathbb{C}, \mathbb{C}). \tag{4.6}$$

The Euler-Lagrange equations corresponding to Lagrange multiplier condition (4.6) are familiar from the theory of incompressible nonlinear elasticity.

Proposition 4.8. *Suppose $u \in \dot W^{1,2}(\mathbb{C}, \mathbb{C})$. Then a function $b \in \mathrm{BMO}(\mathbb{C})$ is a Lagrange multiplier for u if and only if u and b satisfy the equation*

$$u_{z\bar z} = (bu_z)_{\bar z} - (bu_{\bar z})_z \tag{4.7}$$

in the sense of distributions. In real notation (4.7) can be written as

$$\Delta u_1 = 2(\partial_x(b\partial_y u_2) - \partial_y(b\partial_x u_2)), \quad \Delta u_2 = 2(\partial_y(b\partial_x u_1) - \partial_x(b\partial_y u_1)). \tag{4.8}$$

Proof. When $b \in \mathrm{BMO}(\mathbb{C})$, condition (4.7) means that

$$\int^*_{\mathbb{C}} b\mathbf{Re}(u_z\overline{\phi_z} - u_{\bar z}\overline{\phi_{\bar z}}) = \mathbf{Re}\left(\int_{\mathbb{C}} b(u_z\overline{\phi_z} - u_{\bar z}\overline{\phi_{\bar z}})\right) = \mathbf{Re}\int_{\mathbb{C}} u_{\bar z}\overline{\phi_{\bar z}} \tag{4.9}$$

for all $\phi \in C_0^\infty(\mathbb{C}, \mathbb{C})$. Thus (4.6) implies (4.7).

Conversely, suppose (4.7) is valid, i.e., (4.9) holds for all $\phi \in C_0^\infty(\mathbb{C}, \mathbb{C})$. Since $\int^*_{\mathbb{C}} b\,\mathbf{Re}(u_z\overline{\phi_z} - u_{\bar z}\overline{\phi_{\bar z}}) = \int^*_{\mathbb{C}} b(J_{u_1+i\phi_2} + J_{\phi_1+iu_2})$ for all $\phi \in \dot W^{1,2}(\mathbb{C}, \mathbb{C})$, the operator $J\colon \dot W^{1,2}(\mathbb{C}, \mathbb{C}) \to \mathcal{H}^1(\mathbb{C})$ is continuous and $C_0^\infty(\mathbb{C}, \mathbb{C})$ is dense in $\dot W^{1,2}(\mathbb{C}, \mathbb{C})$, we conclude that (4.7) implies (4.6). □

In the next section we discuss what it means in operator theoretical terms for $b \in \mathrm{BMO}(\mathbb{C})$ to be a Lagrange multiplier for $u \in \dot W^{1,2}(\mathbb{C}, \mathbb{C})$.

4.3. **Operator theoretical treatment of Lagrange multipliers.** Recall that we defined in §2.1 the Banach space

$$\dot{W}^{1,2}_{\mathbb{D}}(\mathbb{C},\mathbb{C}) := \left\{ u \in \dot{W}^{1,2}(\mathbb{C},\mathbb{C})\colon \int_{\mathbb{D}} u = 0 \right\};$$

the norm is inherited from $\dot{W}^{1,2}(\mathbb{C},\mathbb{C})$. In this section we study, for a fixed mapping $u \in \dot{W}^{1,2}_{\mathbb{D}}(\mathbb{C},\mathbb{C})$, the linear operator $J'_u\colon \dot{W}^{1,2}_{\mathbb{D}}(\mathbb{C},\mathbb{C}) \to \mathcal{H}^1(\mathbb{C})$ and its transpose $(J'_u)^*\colon \mathrm{BMO}(\mathbb{C}) \to \dot{W}^{1,2}_{\mathbb{D}}(\mathbb{C},\mathbb{C})^*$. We choose $\dot{W}^{1,2}_{\mathbb{D}}(\mathbb{C},\mathbb{C})$ as the domain of J'_u because $\dot{W}^{1,2}(\mathbb{C},\mathbb{C})$ is not a normed space and this restriction of the domain does not affect the range of J'_u.

For the remainder of this section we denote

$$\mathbf{E}[u] := \int_{\mathbb{C}} |u_{\bar{z}}|^2$$

for every $u \in \dot{W}^{1,2}_{\mathbb{D}}(\mathbb{C},\mathbb{C})$ and the Gâteaux derivative of $\mathbf{E}\colon \dot{W}^{1,2}_{\mathbb{D}}(\mathbb{C},\mathbb{C}) \to \mathbb{R}$ by $\mathbf{E}'$. We now fix $u \in \dot{W}^{1,2}_{\mathbb{D}}(\mathbb{C},\mathbb{C}) \setminus \{0\}$.

The Lagrange multiplier condition

$$\langle \mathbf{E}'[u], \phi \rangle = \int_{\mathbb{C}}^{*} b J'_u \phi \quad \text{for all } \phi \in \dot{W}^{1,2}(\mathbb{C},\mathbb{C})$$

can be written as $\mathbf{E}'[u] = (J'_u)^* b$. Thus the existence of a Lagrange multiplier means that $\mathbf{E}'[u] \in \mathrm{ran}(J'_u)^*$. In particular, if the transpose $(J'_u)^*\colon \mathrm{BMO}(\mathbb{C}) \to \dot{W}^{1,2}_{\mathbb{D}}(\mathbb{C},\mathbb{C})^*$ were a surjection, that would imply the existence of a Lagrange multiplier. However, if $(J'_u)^*$ were a surjection, we would have $(\ker J'_u)^{\perp} = \mathrm{ran}(J'_u)^* = \dot{W}^{1,2}_{\mathbb{D}}(\mathbb{C},\mathbb{C})$. Since $J'_u \bar{u} = J_{u_1 - iu_2} + J_{u_1 + iu_2} = 0$ and $\bar{u} \neq 0$, we have $\ker J'_u \supsetneq \{0\}$, and so $(J'_u)^*$ cannot map $\mathrm{BMO}(\mathbb{C})$ onto $\dot{W}^{1,2}_{\mathbb{D}}(\mathbb{C},\mathbb{C})^*$.

Another standard way to construct a Lagrange multiplier would be to show that the range $\mathrm{ran}\, J'_u \subset \mathcal{H}^1(\mathbb{C})$ is closed and that $\mathbf{E}'[u] \in (\ker J'_u)^{\perp}$, that is,

$$J'_u \phi = 0 \implies \langle \mathbf{E}'[u], \phi \rangle = 0. \tag{4.10}$$

Indeed, suppose $\mathrm{ran}\, J'_u$ is closed and (4.10) holds. Since $\mathrm{ran}(J'_u)^*$ is a subspace of the Hilbert space $\dot{W}^{1,2}_{\mathbb{D}}(\mathbb{C},\mathbb{C})$, we have

$$\overline{\mathrm{ran}(J'_u)^*} = (\mathrm{ran}(J'_u)^*)^{\perp\perp} = (\ker J'_u)^{\perp} \tag{4.11}$$

(see [Con90, Corollary I.2.9]). Furthermore, by the Closed range theorem, $\mathrm{ran}(J'_u)^*$ is closed since $\mathrm{ran}\, J'_u$ is. When this is combined with (4.10) and (4.11) we conclude that $\mathbf{E}'[u] \in (\ker J'_u)^{\perp} = \mathrm{ran}(J'_u)^*$ and so u has a Lagrange multiplier.

It is now natural to ask whether the range of the Gâteaux derivative $J'_u\colon \dot{W}^{1,2}_{\mathbb{D}}(\mathbb{C},\mathbb{C}) \to \mathcal{H}^1(\mathbb{C})$ is closed for every $u \in \dot{W}^{1,2}_{\mathbb{D}}(\mathbb{C},\mathbb{C})$ or at least every energy-minimal solution. It turns out that if a Lagrange multiplier exists, then under very non-restrictive conditions on u the range $\mathrm{ran}\, J'_u$ is *not* closed. We omit the proof of this fact as we have no other use for the result.

Even though standard operator theoretical considerations are not sufficient to produce Lagrange multipliers in our setting, in the next two chapters we present a different way to construct a large class of Lagrange multipliers that satisfy the uniform norm estimate $\|b\|_{\mathrm{BMO}} \lesssim 1$.

5. On the Existence of Lagrange Multipliers

We carry out most of the analysis on Lagrange multipliers in terms of the operator $f \mapsto |\mathcal{S}f|^2 - |f|^2$ instead of the Jacobian. Recall that $b \in \mathrm{BMO}_{\mathcal{S}}(\mathbb{C})$ is said to be a Lagrange multiplier for $f \in L^2(\mathbb{C}, \mathbb{C})$ if

$$\mathbf{Re} \int_{\mathbb{C}}^{*} b(\mathcal{S}f\overline{\mathcal{S}\varphi} - f\overline{\varphi}) = \mathbf{Re} \int_{\mathbb{C}} f\overline{\varphi} \quad \text{for all } \varphi \in L^2(\mathbb{C}, \mathbb{C}). \tag{5.1}$$

Also recall from the Introduction and §2.8 the norm defined by

$$\|b\|_{\mathrm{BMO}_{\mathcal{S}}} := \sup_{f \in L^2(\mathbb{C},\mathbb{C}) \setminus \{0\}} \frac{\int_{\mathbb{C}}^{*} b(|\mathcal{S}f|^2 - |f|^2)}{\int_{\mathbb{C}} |f|^2} \tag{5.2}$$

for all $b \in \mathrm{BMO}(\mathbb{C})$.

In this chapter we study the norm $\|\cdot\|_{\mathrm{BMO}_{\mathcal{S}}}$ and the corresponding dual norm $\|\cdot\|_{\mathcal{H}^1_{\mathcal{S}}}$ in $\mathcal{H}^1(\mathbb{C})$. We prove, among other things, that every function on the unit sphere $\mathbb{S}_{\mathrm{VMO}_{\mathcal{S}}}$ is a Lagrange multiplier for some energy-minimal solution. As a byproduct we get a large class $L^2_{\mathcal{S}}(\mathbb{C}, \mathbb{C})$ of energy-minimal solutions that satisfy the norm estimate $\int_{\mathbb{C}} |f|^2 \lesssim \||\mathcal{S}f|^2 - |f|^2\|_{\mathcal{H}^1}$ we seek. This class studied more extensively in §6.

5.1. The norm $\|\cdot\|_{\mathrm{BMO}_{\mathcal{S}}}$ and Lagrange multipliers.

The norms $\|\cdot\|_{\mathrm{BMO}_{\mathcal{S}}}$ and $\|\cdot\|_{\mathcal{H}^1_{\mathcal{S}}}$ allow more precise quantitative analysis of Jacobians and Lagrange multipliers than the norms that are normally used in $\mathrm{VMO}(\mathbb{C})$, $\mathcal{H}^1(\mathbb{C})$ and $\mathrm{BMO}(\mathbb{C})$. As a first example of this we prove the following simple sharp inequality.

Lemma 5.1. *Suppose $f \in L^2(\mathbb{C}, \mathbb{C})$. Then*

$$\||\mathcal{S}f|^2 - |f|^2\|_{\mathcal{H}^1_{\mathcal{S}}} \leq \int_{\mathbb{C}} |f|^2. \tag{5.3}$$

Proof. If $b \in \mathbb{S}_{\mathrm{VMO}_{\mathcal{S}}}$, then the definition of $\|b\|_{\mathrm{BMO}_{\mathcal{S}}}$ implies that

$$\int_{\mathbb{C}}^{*} b(|\mathcal{S}f|^2 - |f|^2) \leq \|b\|_{\mathrm{BMO}_{\mathcal{S}}} \int_{\mathbb{C}} |f|^2 = \int_{\mathbb{C}} |f|^2.$$

Taking supremum over $\mathbb{S}_{\mathrm{VMO}_{\mathcal{S}}}$ yields (5.3). □

We next replicate Proposition 1.13 that connects the norm $\|\cdot\|_{\mathrm{BMO}_{\mathcal{S}}}$ to Lagrange multipliers: $b \in \mathbb{S}_{\mathrm{BMO}_{\mathcal{S}}}$ is a Lagrange multiplier for $f \in L^2(\mathbb{C}, \mathbb{C}) \setminus \{0\}$ if and only if the supremum at (5.2) is achieved at f.

Proposition 5.2. *Suppose $b \in \mathbb{S}_{\mathrm{BMO}_{\mathcal{S}}}$ and $f \in L^2(\mathbb{C}, \mathbb{C})$. The following conditions are equivalent.*

(i) b is a Lagrange multiplier for f.

(ii) $\int_{\mathbb{C}}^{*} b(|\mathcal{S}f|^2 - |f|^2) = \int_{\mathbb{C}} |f|^2$.

If these conditions hold, then f is an energy minimizer.

Proof. If (i) holds, setting $\varphi = f$ in (5.1) yields (ii).

Suppose then (ii) holds and let $\varphi \in L^2(\mathbb{C}, \mathbb{C})$. Since $\|b\|_{\mathrm{BMO}_{\mathcal{S}}} = 1$, we have

$$I(\epsilon) := \int_{\mathbb{C}}^{*} b(|\mathcal{S}(f + \epsilon\varphi)|^2 - |f + \epsilon\varphi|^2) - \int_{\mathbb{C}} |f + \epsilon\varphi|^2 \leq 0$$

for every $\epsilon \in \mathbb{R}$, and by condition (ii), I achieves its maximum when $\epsilon = 0$. Differentiation of I at 0 gives (5.1). Hence, (ii) implies (i).

We now prove that if (i) and (ii) hold, then f is an energy minimizer. Suppose $g \in L^2(\mathbb{C},\mathbb{C})$ satisfies $|\mathcal{S}g|^2 - |g|^2 = |\mathcal{S}f|^2 - |f|^2$. Proposition 5.2, the definition of $\|b\|_{\mathrm{BMO}_\mathcal{S}}$ and the assumption $\|b\|_{\mathrm{BMO}_\mathcal{S}} = 1$ imply that

$$\int_{\mathbb{C}} |f|^2 = \int_{\mathbb{C}}^* b(|\mathcal{S}f|^2 - |f|^2) = \int_{\mathbb{C}}^* b(|\mathcal{S}g|^2 - |g|^2) \leq \int_{\mathbb{C}} |g|^2.$$

This proves that f is an energy minimizer. □

Remark 5.3. In Proposition 5.2 and in most of the results on Lagrange multipliers that follow we assume that $\|b\|_{\mathrm{BMO}_\mathcal{S}} = 1$. This assumption allows the maximization technique of the proof above to work, and it is also necessary in applying isometric Banach space theory to Lagrange multipliers.

In general, if $b \in \mathrm{BMO}_\mathcal{S}(\mathbb{C})$ is a Lagrange multiplier for $f \in L^2(\mathbb{C},\mathbb{C}) \setminus \{0\}$, then $\int_{\mathbb{C}}^* b(|\mathcal{S}f|^2 - |f|^2) = \int_{\mathbb{C}} |f|^2$ and so $\|b\|_{\mathrm{BMO}_\mathcal{S}} \geq 1$. If Lagrange multipliers with $\|b\|_{\mathrm{BMO}_\mathcal{S}} > 1$ exist, it appears that they cannot be found by the methods used in this dissertation.

5.2. **Lagrange multipliers in** $\mathrm{VMO}_\mathcal{S}(\mathbb{C})$**.** The proofs of Theorems 1.14 and 1.16 are started in this section; the former is completed in §6.2 and the latter in §5.3. Recall from the Introduction that we defined

$$L^2_\mathcal{S}(\mathbb{C},\mathbb{C}) := \left\{ f \in L^2(\mathbb{C},\mathbb{C})\colon \int_{\mathbb{C}} |f|^2 = \| |\mathcal{S}f|^2 - |f|^2 \|_{\mathcal{H}^1_\mathcal{S}} \right\}. \tag{5.4}$$

and $\mathbb{S}_{L^2_\mathcal{S}} := L^2_\mathcal{S}(\mathbb{C},\mathbb{C}) \cap \mathbb{S}_{L^2}$. We now begin to study $\mathbb{S}_{L^2_\mathcal{S}}$.

Proposition 5.4. *If $b \in \mathbb{S}_{\mathrm{VMO}_\mathcal{S}}$, then $\int_{\mathbb{C}}^* b(|\mathcal{S}f|^2 - |f|^2) = 1$ for some $f \in \mathbb{S}_{L^2_\mathcal{S}}$.*

Proof. Choose a sequence $(f^j)_{j=1}^\infty$ in $L^2(\mathbb{C},\mathbb{C})$ such that $\int_{\mathbb{C}} |f^j|^2 = 1$ for every $j \in \mathbb{N}$ and

$$\lim_{j\to\infty} \int_{\mathbb{C}}^* b(|\mathcal{S}f^j|^2 - |f^j|^2) = 1. \tag{5.5}$$

The unit ball of $L^2(\mathbb{C},\mathbb{C})$ is weakly compact, and therefore there exists $f \in L^2(\mathbb{C},\mathbb{C})$ such that, moving to a subsequence which we don't relabel, $f^j \rightharpoonup f$ in $L^2(\mathbb{C},\mathbb{C})$. By the weak lower semicontinuity of the norm, $\int_{\mathbb{C}} |f|^2 \leq \liminf_{j\to\infty} \int_{\mathbb{C}} |f^j|^2 = 1$.

Since $b \in \mathrm{VMO}_\mathcal{S}(\mathbb{C})$ and $f^j \rightharpoonup f$, Proposition 2.30 yields

$$\int_{\mathbb{C}}^* b(|\mathcal{S}f|^2 - |f|^2) = \lim_{j\to\infty} \int_{\mathbb{C}}^* b(|\mathcal{S}f^j|^2 - |f^j|^2) = 1. \tag{5.6}$$

By using the assumption $\|b\|_{\mathrm{BMO}_\mathcal{S}} = 1$ and Lemma 5.1 we obtain

$$1 = \int_{\mathbb{C}}^* b(|\mathcal{S}f|^2 - |f|^2) \leq \| |\mathcal{S}f|^2 - |f|^2 \|_{\mathcal{H}^1_\mathcal{S}} \leq \int_{\mathbb{C}} |f|^2 \leq 1,$$

and so $f \in \mathbb{S}_{L^2_\mathcal{S}}$. □

Since $\int_{\mathbb{C}} |f|^2 = 1$ for every $f \in \mathbb{S}_{L^2_\mathcal{S}}$, Lemma 5.4 has the following direct consequence.

Corollary 5.5. *If $b \in \mathbb{S}_{\mathrm{VMO}_\mathcal{S}}$, then b is a Lagrange multiplier for some $f \in \mathbb{S}_{L^2_\mathcal{S}}$.*

Next, recall the following notion from Banach space theory.

Definition 5.6. Let X be a Banach space. A set $B \subset \mathbb{B}_{X^*}$ is called a *James boundary of* X if for every $x \in \mathbb{S}_X$ there exists $x^* \in B$ such that $\langle x^*, x \rangle = 1$.

Lemma 5.4 and Proposition 5.5 have the following equivalent formulation.

Corollary 5.7. *The set* $\{|\mathcal{S}f|^2 - |f|^2 \colon f \in \mathbb{S}_{L^2_{\mathcal{S}}}\}$ *forms a James boundary of* $\mathrm{VMO}_{\mathcal{S}}(\mathbb{C})$.

Since $\mathcal{H}^1_{\mathcal{S}}(\mathbb{C})$ is separable by Proposition 2.15, Corollary 5.7 allows us to use the following special case of a theorem of G. Godefroy (see [FHHMPZ01, Theorem 3.46]).

Theorem 5.8. *Let X be a Banach space. If $B \subset \mathbb{B}_{X^*}$ is a separable James boundary for X, then the closed convex hull $\overline{\mathrm{co}}(B) = \mathbb{B}_{X^*}$.*

Corollary 5.7 and Theorem 5.8 have the following consequence.

Theorem 5.9. $\overline{\mathrm{co}}(\{|\mathcal{S}f|^2 - |f|^2 \colon f \in \mathbb{S}_{L^2_{\mathcal{S}}}\}) = \mathbb{B}_{\mathcal{H}^1_{\mathcal{S}}}$.

Theorem 5.9 plays a crucial role in the next section.

5.3. **Characterizations of the norms $\|\cdot\|_{\mathrm{BMO}_{\mathcal{S}}}$ and $\|\cdot\|_{\mathcal{H}^1_{\mathcal{S}}}$.** Proposition 5.5 raises the question under what conditions an energy minimizer $f \in L^2(\mathbb{C}, \mathbb{C}) \setminus \{0\}$ has a Lagrange multiplier. The main goal of this section is to prove Theorem 1.16 which says that a mapping $f \in L^2(\mathbb{C}, \mathbb{C})$ has a Lagrange multiplier $b \in \mathbb{S}_{\mathrm{BMO}_{\mathcal{S}}}$ if and only if $\int_{\mathbb{C}} |f|^2 = \||\mathcal{S}f|^2 - |f|^2\|_{\mathcal{H}^1_{\mathcal{S}}}$. The proof is based on the following result. We denote the dual norm of $\mathrm{BMO}_{\mathcal{S}}(\mathbb{C})$ by $\|\cdot\|_{(\mathrm{VMO}_{\mathcal{S}})^{**}}$.

Theorem 5.10. *If $b \in \mathrm{BMO}_{\mathcal{S}}(\mathbb{C})$, then*

$$\|b\|_{\mathrm{BMO}_{\mathcal{S}}} = \|b\|_{(\mathrm{VMO}_{\mathcal{S}})^{**}} = \sup_{f \in \mathbb{S}_{L^2_{\mathcal{S}}}} \int_{\mathbb{C}}^{*} b(|\mathcal{S}f|^2 - |f|^2).$$

Proof. Let $b \in \mathrm{BMO}_{\mathcal{S}}(\mathbb{C})$. By Lemma 5.1,

$$\begin{aligned} \|b\|_{\mathrm{BMO}_{\mathcal{S}}} &= \sup_{\int_{\mathbb{C}} |f|^2 \le 1} \int_{\mathbb{C}}^{*} b(|\mathcal{S}f|^2 - |f|^2) \le \sup_{\||\mathcal{S}f|^2 - |f|^2\|_{\mathcal{H}^1_{\mathcal{S}}} \le 1} \int_{\mathbb{C}}^{*} b(|\mathcal{S}f|^2 - |f|^2) \\ &\le \sup_{\|h\|_{\mathcal{H}^1_{\mathcal{S}}} \le 1} \int_{\mathbb{C}}^{*} bh = \|b\|_{\mathrm{VMO}_{\mathcal{S}}^{**}}. \end{aligned}$$

The inequality $\sup_{f \in \mathbb{S}_{L^2_{\mathcal{S}}}} \int_{\mathbb{C}}^{*} b(|\mathcal{S}f|^2 - |f|^2) \le \|b\|_{\mathrm{BMO}_{\mathcal{S}}}$ is obvious.

In order to complete the proof note that, by Theorem 5.9,

$$\|b\|_{(\mathrm{VMO}_{\mathcal{S}})^{**}} = \sup_{h \in \mathbb{B}_{\mathcal{H}^1_{\mathcal{S}}}} \int_{\mathbb{C}}^{*} bh = \sup_{h \in \overline{\mathrm{co}}(\{|\mathcal{S}f|^2 - |f|^2 \colon f \in \mathbb{S}_{L^2_{\mathcal{S}}}\})} \int_{\mathbb{C}}^{*} bh.$$

It is a standard fact about linear functionals that

$$\sup_{h \in \overline{\mathrm{co}}(\{|\mathcal{S}f|^2 - |f|^2 \colon f \in \mathbb{S}_{L^2_{\mathcal{S}}}\})} \int_{\mathbb{C}}^{*} bh = \sup_{h \in \{|\mathcal{S}f|^2 - |f|^2 \colon f \in \mathbb{S}_{L^2_{\mathcal{S}}}\}} \int_{\mathbb{C}}^{*} bh = \sup_{f \in \mathbb{S}_{L^2_{\mathcal{S}}}} \int_{\mathbb{C}}^{*} b(|\mathcal{S}f|^2 - |f|^2).$$

□

Theorem 5.10 shows, in particular, that $\mathrm{BMO}_{\mathcal{S}}(\mathbb{C})$ is *isometrically* the bidual of $\mathrm{VMO}_{\mathcal{S}}(\mathbb{C})$ and

$$\int_{\mathbb{C}}^{*} bh \le \|b\|_{\mathrm{BMO}_{\mathcal{S}}} \|h\|_{\mathcal{H}^1_{\mathcal{S}}} \quad \text{for all } b \in \mathrm{BMO}_{\mathcal{S}}(\mathbb{C}) \text{ and } h \in \mathcal{H}^1_{\mathcal{S}}(\mathbb{C}). \tag{5.7}$$

We next show that Theorem 5.10 implies Theorem 1.16.

Proof of Theorem 1.16. Suppose $f \in L^2(\mathbb{C},\mathbb{C})$. If f has a Lagrange multiplier $b \in \mathbb{S}_{\mathrm{BMO}_\mathcal{S}}$, then Proposition 5.2, (5.7) and the assumption $\|b\|_{\mathrm{BMO}_\mathcal{S}} = 1$ imply that

$$\int_{\mathbb{C}} |f|^2 = \int_{\mathbb{C}}^* b(|\mathcal{S}f|^2 - |f|^2) \leq \| |\mathcal{S}f|^2 - |f|^2 \|_{\mathcal{H}^1_\mathcal{S}}.$$

Lemma 5.1 gives the converse inequality, and so $\| |\mathcal{S}f|^2 - |f|^2 \|_{\mathcal{H}^1_\mathcal{S}} = \int_{\mathbb{C}} |f|^2$.

Conversely, if $\| |\mathcal{S}f|^2 - |f|^2 \|_{\mathcal{H}^1_\mathcal{S}} = \int_{\mathbb{C}} |f|^2$, then Theorem 5.10 and the Hahn-Banach theorem imply that there exists $b \in \mathbb{S}_{\mathrm{BMO}_\mathcal{S}}$ such that

$$\int_{\mathbb{C}}^* b(|\mathcal{S}f|^2 - |f|^2) = \| |\mathcal{S}f|^2 - |f|^2 \|_{\mathcal{H}^1_\mathcal{S}} = \int_{\mathbb{C}} |f|^2.$$

By Proposition 5.2, b is a Lagrange multiplier for f.

The Hahn-Banach theorem implies that every $b \in \mathbb{S}_{\mathrm{VMO}_\mathcal{S}}$, when considered as an element of $\mathrm{BMO}_\mathcal{S}(\mathbb{C})$, is norm-attaining (see e.g. [Con90, p. 79]). □

We continue the study of the geometry of the spaces $\mathrm{VMO}_\mathcal{S}(\mathbb{C})$, $\mathcal{H}^1_\mathcal{S}(\mathbb{C})$ and $\mathrm{BMO}_\mathcal{S}(\mathbb{C})$ in §6.

5.4. **Connection to commutators.** As mentioned in the Introduction, Lagrange multipliers turn out to be closely related to commutators $\mathcal{S}b - b\mathcal{S}$, where $\mathcal{S}$ is the Beurling transform and $b \in \mathrm{BMO}(\mathbb{C})$. We will make use of the operator $K_b := \overline{(\mathcal{S}b - b\mathcal{S})\overline{\mathcal{S}\cdot}} \colon L^2(\mathbb{C},\mathbb{C}) \to L^2(\mathbb{C},\mathbb{C})$ defined in §2.8. We now present a proof of Proposition 1.23 which we recall below.

Proposition 5.11. *Suppose $b \in \mathrm{BMO}(\mathbb{C})$ and $f \in L^2(\mathbb{C},\mathbb{C})$. The following conditions are equivalent.*

(i) $f \in \ker(I - K_b)$.

(ii) b is a Lagrange multiplier for f.

Furthermore, if $b \in \mathbb{S}_{\mathrm{VMO}_\mathcal{S}}$, then $\ker(I - K_b)$ is finite-dimensional and contained in $L^2_\mathcal{S}(\mathbb{C},\mathbb{C})$.

Proof. We shall prove that

$$\int_{\mathbb{C}}^* b\mathbf{Re}(\mathcal{S}f\overline{\mathcal{S}\varphi} - f\overline{\varphi}) = \mathbf{Re}\int_{\mathbb{C}} (K_b f)\overline{\varphi}$$

for every $\varphi \in L^2(\mathbb{C},\mathbb{C})$. Then the condition $f \in \ker(I - K_b)$, that is, equation

$$K_b f = f, \tag{5.8}$$

is clearly equivalent to Lagrange multiplier condition $\int_{\mathbb{C}}^* b\mathbf{Re}(\mathcal{S}f\overline{\mathcal{S}\varphi} - f\overline{\varphi}) = \mathbf{Re}\int_{\mathbb{C}} f\overline{\varphi}$.

Let $\varphi \in L^2(\mathbb{C},\mathbb{C})$ and use (2.14) and the self-adjointness of K_b to write

$$\begin{aligned}
&2\int_{\mathbb{C}}^* b\mathbf{Re}(\mathcal{S}f\overline{\mathcal{S}\varphi} - f\overline{\varphi}) \\
=\ & \int_{\mathbb{C}}^* b(|\mathcal{S}(f+\varphi)|^2 - |f+\varphi|^2) - \int_{\mathbb{C}}^* b(|\mathcal{S}f|^2 - |f|^2) - \int_{\mathbb{C}}^* b(|\mathcal{S}\varphi|^2 - |\varphi|^2) \\
=\ & \int_{\mathbb{C}} (f+\varphi)\overline{K_b(f+\varphi)} - \int_{\mathbb{C}} f\overline{K_b f} - \int_{\mathbb{C}} \varphi\overline{K_b\varphi} \\
=\ & \int_{\mathbb{C}} (f\overline{K_b\varphi} + \varphi\overline{K_b f}) = \int_{\mathbb{C}} ((K_b f)\overline{\varphi} + \overline{K_b f}\varphi) = 2\mathbf{Re}\int_{\mathbb{C}} (K_b f)\overline{\varphi}.
\end{aligned}$$

Suppose now that $b \in \mathbb{S}_{\mathrm{VMO}_\mathcal{S}}$. By Lemma 2.34, K_b is a compact operator. The finite-dimensionality of $\ker(I - K_b)$ now follows by elementary Hilbert space geometry (see [Con90, Proposition II.4.13]). Furthermore, Theorem 1.16 and the equivalence of conditions (i) and (ii) imply that $\ker(I - K_b) \subset L^2_\mathcal{S}(\mathbb{C}, \mathbb{C})$. □

Note that when $b \in \mathbb{S}_{\mathrm{BMO}_\mathcal{S}}$, Proposition 2.35 implies that (5.8) can be written as $K_b f = \|K_b\|_{L^2 \to L^2} f$, that is, f is an eigenvector corresponding to the maximal eigenvalue $\|K_b\|_{L^2 \to L^2}$.

6. On the Class $\mathbb{S}_{L^2_{\mathcal{S}}}$

As we have seen in the Introduction and §5, the geometry of the unit spheres $\mathbb{S}_{\mathrm{VMO}_{\mathcal{S}}}$, $\mathbb{S}_{\mathcal{H}^1_{\mathcal{S}}}$, $\mathbb{S}_{\mathrm{BMO}_{\mathcal{S}}}$ and $\mathbb{S}_{L^2}$ is crucial in the study of Lagrange multipliers. Motivated, among other things, by Theorem 1.16 we are particularly interested in the properties of the set

$$\mathbb{S}_{L^2_{\mathcal{S}}} = \left\{ f \in L^2(\mathbb{C},\mathbb{C})\colon \int_{\mathbb{C}} |f|^2 = \| |\mathcal{S}f|^2 - |f|^2 \|_{\mathcal{H}^1_{\mathcal{S}}} = 1 \right\}. \tag{6.1}$$

In this chapter we prove Theorem 1.19 which we restate below.

Theorem 6.1. *The following statements hold.*

(i) $\{|\mathcal{S}f|^2 - |f|^2 \colon f \in \mathbb{S}_{L^2_{\mathcal{S}}}\}$ *contains all the extreme points of* $\mathbb{B}_{\mathcal{H}^1_{\mathcal{S}}}$.

(ii) $\{|\mathcal{S}f|^2 - |f|^2 \colon f \in \mathbb{S}_{L^2_{\mathcal{S}}}\}$ *is closed in the relative weak-$*$ topology of* $\mathbb{S}_{\mathcal{H}^1_{\mathcal{S}}}$. *In particular,* $\{|\mathcal{S}f|^2 - |f|^2 \colon f \in \mathbb{S}_{L^2_{\mathcal{S}}}\}$ *is norm closed.*

In §6.2 we use Theorem 6.1 to prove Theorem 1.14 which we also restate here.

Theorem 6.2. *Let* $b \in \mathbb{S}_{\mathrm{BMO}_{\mathcal{S}}}$. *The following conditions are equivalent.*

(i) b *is a Lagrange multiplier for some* $f \in L^2(\mathbb{C},\mathbb{C}) \setminus \{0\}$.

(ii) b *is norm-attaining.*

In particular, every $b \in \mathbb{S}_{\mathrm{VMO}_{\mathcal{S}}}$ *is a Lagrange multiplier for some* $f \in L^2(\mathbb{C},\mathbb{C}) \setminus \{0\}$.

This chapter culminates in §6.3 in the proof of the main result of this dissertation, Theorem 1.27.

6.1. Topological properties of $\{|\mathcal{S}f|^2 - |f|^2 \colon f \in \mathbb{S}_{L^2_{\mathcal{S}}}\}$.

The topological properties of $L^2(\mathbb{C},\mathbb{C})$ can be capitalized on when studying the set $\{|\mathcal{S}f|^2 - |f|^2 \colon f \in \mathbb{S}_{L^2_{\mathcal{S}}}\}$. As an instance of this, since $L^2(\mathbb{C},\mathbb{C})$ is a Hilbert space, a sequence converges if it converges weakly and the norms of the elements of the sequence converge. Another feature useful to us is the weak compactness of the unit ball $\mathbb{B}_{L^2}$. These properties are next used to prove the second of the two claims of Theorem 6.1.

Proof of Theorem 6.1(ii). Since $\mathrm{VMO}_{\mathcal{S}}(\mathbb{C})$ is separable, the unit ball $\mathbb{B}_{\mathcal{H}^1_{\mathcal{S}}}$ is metrizable in the weak-$*$ topology, and so it suffices to consider sequences instead of nets. Suppose $f^j \in \mathbb{S}_{L^2_{\mathcal{S}}}$ for every $j \in \mathbb{N}$ and $|\mathcal{S}f^j|^2 - |f^j|^2 \overset{*}{\rightharpoonup} h \in \mathbb{S}_{\mathcal{H}^1_{\mathcal{S}}}$. Since $\int_{\mathbb{C}} |f^j|^2 = 1$ for every $j \in \mathbb{N}$, we may pass to a subsequence and find $f \in L^2(\mathbb{C},\mathbb{C})$ such that $f^j \rightharpoonup f$. By Proposition 2.30, $|\mathcal{S}f^j|^2 - |f^j|^2 \overset{*}{\rightharpoonup} |\mathcal{S}f|^2 - |f|^2$, and so $h = |\mathcal{S}f|^2 - |f|^2$. By Lemma 5.1,

$$1 = \|h\|_{\mathcal{H}^1_{\mathcal{S}}} = \| |\mathcal{S}f|^2 - |f|^2 \|_{\mathcal{H}^1_{\mathcal{S}}} \leq \int_{\mathbb{C}} |f|^2 \leq \liminf_{j\to\infty} \int_{\mathbb{C}} |f^j|^2 = 1, \tag{6.2}$$

and therefore $f \in \mathbb{S}_{L^2_{\mathcal{S}}}$. □

The coincidence of the norm and relative weak topologies in $\mathbb{S}_{L^2}$ is reflected in the following result.

Proposition 6.3. *The relative weak-$*$ topology of* $\mathbb{S}_{\mathcal{H}^1_{\mathcal{S}}}$ *and norm topology coincide in* $\{|\mathcal{S}f|^2 - |f|^2 \colon f \in \mathbb{S}_{L^2_{\mathcal{S}}}\}$.

Proof. As in the previous proof, it suffices to consider sequences. Let $f^1, f^2, \dots \in \mathbb{S}_{L^2_{\mathcal{S}}}$ and $f \in \mathbb{S}_{L^2_{\mathcal{S}}}$, and suppose $|\mathcal{S}f^j|^2 - |f^j|^2 \overset{*}{\rightharpoonup} |\mathcal{S}f|^2 - |f|^2$. We will prove the claim by showing that every subsequence of $(|\mathcal{S}f^j|^2 - |f^j|^2)_{j=1}^\infty$ has a subsequence that converges in norm to $|\mathcal{S}f|^2 - |f|^2$. Equivalently, we work with the sequence $(f^j)_{j=1}^\infty$. Fix a subsequence and use the weak compactness of $\mathbb{B}_{L^2}$ to pass to a further subsequence that converges weakly, $f^j \rightharpoonup g$. By Lemma 2.30, $|\mathcal{S}f^j|^2 - |f^j|^2 \overset{*}{\rightharpoonup} |\mathcal{S}g|^2 - |g|^2$. On the other hand, we assumed that $|\mathcal{S}f^j|^2 - |f^j|^2 \overset{*}{\rightharpoonup} |\mathcal{S}f|^2 - |f|^2$, and so $|\mathcal{S}f|^2 - |f|^2 = |\mathcal{S}g|^2 - |g|^2$. As in (6.2), $\|g\|_{L^2} = 1$. Now $f^j \rightharpoonup g$ and $1 = \|f^j\|_{L^2} \to \|g\|_{L^2}$ imply that $\|f^j - g\|_{L^2} \to 0$. By Corollary 2.19, $|\mathcal{S}f^j|^2 - |f^j|^2 \to |\mathcal{S}g|^2 - |g|^2 = |\mathcal{S}f|^2 - |f|^2$ in $\mathcal{H}^1_{\mathcal{S}}(\mathbb{C})$. □

6.2. **Extreme points of $\mathbb{B}_{\mathcal{H}^1_{\mathcal{S}}}$.** The purpose of this section is to prove Theorem 6.1(i) and Theorem 1.14. We will use the following theorem of M. Milman (see [Meg98, Theorem 2.10.15]).

Theorem 6.4. *Let K be a nonempty compact subset of a Hausdorff locally convex space X such that $\overline{\mathrm{co}}(K)$ is compact. Then every extreme point of $\overline{\mathrm{co}}(K)$ lies in K.*

We now complete the proof of Theorem 6.1 by showing that the set $\{|\mathcal{S}f|^2 - |f|^2 \colon f \in \mathbb{S}_{L^2_{\mathcal{S}}}\}$ contains all the extreme points of $\mathbb{B}_{\mathcal{H}^1_{\mathcal{S}}}$.

Proof of Theorem 6.1(i). In Milman's theorem choose X to be $\mathcal{H}^1_{\mathcal{S}}(\mathbb{C})$ endowed with the weak-$*$ topology and let K be the weak-$*$ closure of $\{|\mathcal{S}f|^2 - |f|^2 \colon f \in \mathbb{S}_{L^2_{\mathcal{S}}}\}$ in $\mathcal{H}^1_{\mathcal{S}}(\mathbb{C})$. By Theorem 5.9, $\overline{\mathrm{co}}(K) = \mathbb{B}_{\mathcal{H}^1_{\mathcal{S}}}$, and in particular, the weak-$*$ closed convex hull $\overline{\mathrm{co}}^{\,\mathrm{w}*}(K) = \mathbb{B}_{\mathcal{H}^1_{\mathcal{S}}}$. Since $\mathbb{B}_{\mathcal{H}^1_{\mathcal{S}}}$ is weak-$*$ compact, Milman's theorem implies that every extreme point of $\mathbb{B}_{\mathcal{H}^1_{\mathcal{S}}}$ lies in K.

Let $h \in K$ be an extreme point of $\mathbb{B}_{\mathcal{H}^1_{\mathcal{S}}}$. By the definition of K, there exist $f^j \in \mathbb{S}_{L^2_{\mathcal{S}}}$ such that $|\mathcal{S}f^j|^2 - |f^j|^2 \overset{*}{\rightharpoonup} h$. On the other hand, as an extreme point of $\mathbb{B}_{\mathcal{H}^1_{\mathcal{S}}}$ the function h belongs to $\mathbb{S}_{\mathcal{H}^1_{\mathcal{S}}}$. By Theorem 6.1(ii), $h = |\mathcal{S}f|^2 - |f|^2$ for some $f \in \mathbb{S}_{L^2_{\mathcal{S}}}$. □

Theorem 6.1 is next used to prove Theorem 1.14. Here it is crucial that extreme points of $\mathbb{B}_{\mathcal{H}^1_{\mathcal{S}}}$ exist in abundance (see Corollary 2.43).

Proof of Theorem 1.14. Suppose first $b \in \mathbb{S}_{\mathrm{BMO}_{\mathcal{S}}}$ is a Lagrange multiplier for $f \in L^2(\mathbb{C}, \mathbb{C}) \setminus \{0\}$. By Theorem 1.16 and scaling, we may assume that $f \in \mathbb{S}_{L^2_{\mathcal{S}}}$, and then Proposition 1.13 implies that b attains its norm at $|\mathcal{S}f|^2 - |f|^2 \in \mathbb{S}_{\mathcal{H}^1_{\mathcal{S}}}$.

Conversely, suppose $b \in \mathbb{S}_{\mathrm{BMO}_{\mathcal{S}}}$ is norm-attaining. The set

$$K := \left\{ h \in \mathbb{S}_{\mathcal{H}^1_{\mathcal{S}}} \colon \int_{\mathbb{C}}^* bh = 1 \right\}$$

is non-empty. Since K is closed, convex and bounded, it follows from Corollary 2.43 that K contains an extreme point h. The definition of K easily implies that h is also an extreme point of $\mathbb{B}_{\mathcal{H}^1_{\mathcal{S}}}$. Theorem 6.1 now implies that $h = |\mathcal{S}f|^2 - |f|^2$ for some $f \in \mathbb{S}_{L^2_{\mathcal{S}}}$. By Proposition 1.13, b is a Lagrange multiplier for f. □

6.3. **The duality mapping $D \colon \mathbb{S}_{\mathrm{VMO}_{\mathcal{S}}} \to \mathbb{S}_{\mathcal{H}^1_{\mathcal{S}}}$.** Suppose $b \in \mathbb{S}_{\mathrm{VMO}_{\mathcal{S}}}$. One of the useful features of the set $K := \{h \in \mathbb{S}_{\mathcal{H}^1_{\mathcal{S}}} \colon \int_{\mathbb{C}}^* bh = 1\}$ is that $h \in K$ is an extreme point of K if and only if h is an extreme point of $\mathbb{S}_{\mathcal{H}^1_{\mathcal{S}}}$. Furthermore, K is closed, bounded and convex and therefore K is the closed convex hull of its extreme points (see Corollary 2.43). This allows us to take advantage of Theorem 6.1(i) and motivates study of the duality mapping.

Definition 6.5. Let X be a Banach space. The *duality mapping* $D\colon \mathbb{S}_X \to \mathcal{P}(\mathbb{S}_{X^*})$ is defined by

$$D(x) := \{x^* \in \mathbb{S}_{X^*} : \langle x^*, x\rangle = 1\}.$$

Before studying $D\colon \mathbb{S}_{\mathrm{VMO}_\mathcal{S}} \to \mathbb{S}_{\mathcal{H}^1_\mathcal{S}}$ recall from Propositions 1.13 and 1.23 that when $b \in \mathbb{S}_{\mathrm{VMO}_\mathcal{S}}$ and $f \in \mathbb{S}_{L^2_\mathcal{S}}$, we have $|\mathcal{S}f|^2 - |f|^2 \in D(b)$ if and only if f belongs to the unit circle of the *finite-dimensional* subspace $\ker(I - K_b) \subset L^2(\mathbb{C}, \mathbb{C})$.

We recall Theorem 1.27 for the convenience of the reader.

Theorem 6.6. *Let $b \in \mathbb{S}_{\mathrm{VMO}_\mathcal{S}}$. The following statements hold.*
(i) $D(b)$ is contained in a finite-dimensional subspace of $\mathcal{H}^1_\mathcal{S}(\mathbb{C})$.
(ii) $f \mapsto |\mathcal{S}f|^2 - |f|^2 \colon \ker(I - K_b) \cap \mathbb{S}_{L^2_\mathcal{S}} \to D(b)$ is Lipschitz continuous.
(iii) $\{|\mathcal{S}f|^2 - |f|^2 \colon f \in \ker(I - K_b) \cap \mathbb{S}_{L^2_\mathcal{S}}\}$ contains all the extreme points of $D(b)$.
(iv) $\{|\mathcal{S}f|^2 - |f|^2 \colon f \in \ker(I - K_b) \cap \mathbb{S}_{L^2_\mathcal{S}}\}$ is closed and path-connected.

Proof. Fix $b \in \mathbb{S}_{\mathrm{VMO}_\mathcal{S}}$. Claim ($ii$) follows from Corollary 2.19.

We next prove claim (i). By Corollary 2.43, $D(b)$ is the closed convex hull of its extreme points. All the extreme points of $D(b)$ are also extreme points of $\mathbb{B}_{\mathcal{H}^1_\mathcal{S}}$, and by Theorem 6.1 they belong to $\{|\mathcal{S}f|^2 - |f|^2 \colon f \in \mathbb{S}_{L^2_\mathcal{S}}\}$. We will show that $\{|\mathcal{S}f|^2 - |f|^2 \colon f \in \mathbb{S}_{L^2_\mathcal{S}}\} \cap D(b)$ is contained in a finite-dimensional subspace of $\mathcal{H}^1_\mathcal{S}(\mathbb{C})$; then $D(b)$ has the same property.

Choose an orthonormal basis $\{f^1, \ldots, f^n\}$ for the finite-dimensional complex vector space $\ker(I - K_b)$. If $f \in \mathbb{S}_{L^2_\mathcal{S}}$ and $|\mathcal{S}f|^2 - |f|^2 \in D(b)$, then $f \in \ker(I - K_b)$ and so f is a complex linear combination of $f^1, \ldots, f^n$. Thus $|\mathcal{S}f|^2 - |f|^2$ is a real linear combination of mappings of the form $\mathbf{Re}(\mathcal{S}f^j\overline{\mathcal{S}f^k} - f^j\overline{f^k})$ or $\mathbf{Re}(i(\mathcal{S}f^j\overline{\mathcal{S}f^k} - f^j\overline{f^k}))$ where $1 \le j, k \le n$. Consequently,

$$D(b) \subset \operatorname{span} \bigcup_{j,k=1}^{n} \left\{ \mathbf{Re}(\mathcal{S}f^j\overline{\mathcal{S}f^k} - f^j\overline{f^k}), \mathbf{Im}(\mathcal{S}f^j\overline{\mathcal{S}f^k} - f^j\overline{f^k}) \right\}.$$

This proves claim (i).

Claim (iii) follows from Theorem 6.1(i) and the fact that $h \in D(b)$ is an extreme point of $D(b)$ if and only if it is an extreme point of $\mathbb{B}_{\mathcal{H}^1_\mathcal{S}}$.

Since $\ker(I - K_b) \cap \mathbb{S}_{L^2_\mathcal{S}}$ is compact and path-connected, so is the image set $\{|\mathcal{S}f|^2 - |f|^2 \colon f \in \ker(I - K_b) \cap \mathbb{S}_{L^2_\mathcal{S}}\}$, by continuity. This completes the proof the theorem. ☐

In the Introduction we posed the question whether $f \mapsto |\mathcal{S}f|^2 - |f|^2$ maps $\ker(I - K_b) \cap \mathbb{S}_{L^2_\mathcal{S}}$ onto $D(b)$ for every $b \in \mathbb{S}_{\mathrm{VMO}_\mathcal{S}}$. The following proposition gives a simple partial result.

Proposition 6.7. *For a dense subset of $\mathbb{S}_{\mathrm{VMO}_\mathcal{S}}$ the operator $f \mapsto |\mathcal{S}f|^2 - |f|^2$ is a surjection from $\ker(I - K_b) \cap \mathbb{S}_{L^2_\mathcal{S}}$ to $D(b)$.*

Proof. Since $\mathrm{VMO}_\mathcal{S}(\mathbb{C})$ has the separable dual $\mathcal{H}^1_\mathcal{S}(\mathbb{C})$, the norm $\|\cdot\|_{\mathrm{BMO}_\mathcal{S}}$ is Gâteaux differentiable (even Fréchet differentiable) in a dense subset of $\mathbb{S}_{\mathrm{VMO}_\mathcal{S}}$ (see [FHHMPZ01, Theorem 8.21]). By a result of S. Banach (see [Meg98, Theorem 5.4.17]), when $b \in \mathbb{S}_{\mathrm{VMO}_\mathcal{S}}$, the set $D(b)$ is a singleton if and only if the norm $\|\cdot\|_{\mathrm{BMO}_\mathcal{S}}$ is Gâteaux differentiable at b. When $D(b) = \{h\}$ is a singleton, Theorem 6.6(*iii*) implies that $h = |\mathcal{S}f|^2 - |f|^2$ for some $f \in \mathbb{S}_{L^2_{\mathcal{S}}}$. $\square$

It is natural to ask whether the norm $\|\cdot\|_{\mathrm{BMO}_{\mathcal{S}}}$ is Gâteaux differentiable at *every* point of $\mathbb{S}_{\mathrm{VMO}_{\mathcal{S}}}$. This would, by Theorem 6.6(*iii*), imply a positive answer to Question 1.24 and thereby to Question 1.2. Since $\mathrm{VMO}_{\mathcal{S}}(\mathbb{C})$ is separable, there exists a Gâteaux differentiable norm in $\mathrm{VMO}_{\mathcal{S}}(\mathbb{C})$ (see [FHHMPZ01, Theorem 8.13]).

We also present the following question related to Theorem 1.27.

Question 6.8. *Let* $b \in \mathbb{S}_{\mathrm{VMO}_{\mathcal{S}}}$. *If* $f, g \in \ker(I - K_b)$ *satisfy* $\mathbf{Re} \int_{\mathbb{C}} f\overline{g} = 0$, *does it follow that* $\mathbf{Re}(\mathcal{S}f\overline{\mathcal{S}g} - f\overline{g}) = 0$*?*

Remark 6.9. A positive answer to Question 6.8 would imply a positive answer to Question 1.24. We omit the proof of this fairly standard fact.

It is also tempting to ask whether every element of $D(b)$ is a convex combination of *two* elements of $\{|\mathcal{S}f|^2 - |f|^2 \colon f \in \ker(I - K_b) \cap \mathbb{S}_{L^2_{\mathcal{S}}}\}$. When combined with Theorem 6.1(*ii*), this would imply that every element of $\mathcal{H}^1(\mathbb{C})$ is a sum of two Jacobians.

7. Local Study of Energy-Minimal Solutions and Lagrange Multipliers

Question 1.24 motivates the study of the way an energy minimizer $u \in \dot{W}^{1,2}(\mathbb{C},\mathbb{C})$ (or $f \in L^2(\mathbb{C},\mathbb{C})$) and its Lagrange multiplier $b \in \mathbb{S}_{\mathrm{BMO}_{\mathcal{S}}}$ (if one exists) are related. We present several results on this topic in this chapter and the next one. Working with mappings in $\dot{W}^{1,2}(\mathbb{C},\mathbb{C})$ instead of $L^2(\mathbb{C},\mathbb{C})$ appears to be the most natural option in this context.

Our general strategy can be outlined as follows. If $u \in \dot{W}^{1,2}(\mathbb{C},\mathbb{C})$ has integrable distortion and positive Jacobian in a domain Ω, it follows from Corollary 2.8 that u is a local homeomorphism outside a discrete set. We then restrict to a subdomain where u is a homeomorphism and use the Euler-Lagrange equations given in Proposition 4.8 to gather information about u and the Lagrange multiplier $b \in \mathrm{BMO}(\mathbb{C})$ in the subdomain. In some cases we can then draw conclusions on the behavior of u and b in the whole domain Ω.

7.1. Lagrange multipliers in a bounded domain.

When an energy-minimal solution has a positive Jacobian in a domain, ideas from the mathematical theory of incompressible elasticity can be used to study the solution. For results in this direction relevant to our study see e.g. [Bal77], [LO81], [BOP92], [Le Dre85], [CK09], [Kar12] and the references contained therein. The setting of [CK09] and [Kar12] is mathematically particularly close to ours.

The aim of this section is to give a local representation theorem for a Lagrange multiplier provided it exists. The idea behind the proof of the theorem dates back at least to [LO81] and is used under hypotheses more similar to ours in [CK09]. Some technicalities are, however, needed in order to establish the result in our setting, and we therefore present a proof. Theorem 7.1 applies, in particular, if $b \in \mathrm{BMO}(\mathbb{C})$ is a Lagrange multiplier for $u \in \dot{W}^{1,2}(\mathbb{C},\mathbb{C})$. When $\phi \in W_0^{1,2}(\Omega,\mathbb{C})$, we identify ϕ and the mapping that is obtained by extending ϕ as zero outside Ω.

Theorem 7.1. *Let $b \in \mathrm{BMO}(\mathbb{C})$ and $u \in \dot{W}^{1,2}(\mathbb{C},\mathbb{C})$. Assume u satisfies $J_u > 0$ a.e. in a domain $\Omega \subset \mathbb{C}$ and $u\colon \Omega \to u(\Omega)$ is a homeomorphism with inverse $v := u^{-1}$. If b and u satisfy*

$$\int_{\mathbb{C}}^{*} b\,\mathbf{Re}(u_z\overline{\phi_z} - u_{\bar z}\overline{\phi_{\bar z}}) = \mathbf{Re}\int_{\mathbb{C}} u_{\bar z}\overline{\phi_{\bar z}} \quad \text{for all } \phi \in W_0^{1,2}(\Omega,\mathbb{C}), \tag{7.1}$$

then $b|_\Omega$ is of the form

$$b = q \circ u,$$

where $q \in L^1_{loc}(u(\Omega))$ satisfies

$$q_\zeta = \left(\frac{|v_{\bar\zeta}|^2}{J_v}\right)_\zeta - \left(\frac{v_\zeta\overline{v_{\bar\zeta}}}{J_v}\right)_{\bar\zeta} \tag{7.2}$$

in $u(\Omega)$ in the sense of distributions.

As part of the proof of Theorem 7.1 we show that $|v_{\bar\zeta}|^2/J_v$ and $v_\zeta\overline{v_{\bar\zeta}}/J_v$ are integrable in $u(\Omega)$, so that (7.2) is well-defined. Before presenting the proof we need some definitions and lemmas.

Definition 7.2. Let $\Omega, \Omega' \subset \mathbb{C}$ and let $u\colon \Omega \to \Omega'$ be measurable. The mapping u satisfies *Lusin's condition* $\mathcal{N}$ if for every $E \subset \Omega$,

$$|E| = 0 \implies |u(E)| = 0.$$

When $u\colon \Omega \to \Omega'$ is a bijection, we write Lusin's condition $\mathcal{N}$ for u^{-1} as

$$|u(E)| = 0 \implies |E| = 0 \tag{7.3}$$

for all $E \subset \Omega$.

Lemma 7.3. *Suppose $\Omega \subset \mathbb{C}$ is open, $u \in W^{1,2}(\Omega, u(\Omega))$ is a homeomorphism and $J_u > 0$ a.e. Then both u and u^{-1} satisfy Lusin's condition $\mathcal{N}$.*

Proof. For a proof of the fact that u satisfies Lusin's condition $\mathcal{N}$ see [AIM09, Theorem 3.3.7].

We sketch the proof of condition $\mathcal{N}$ for u^{-1}. The primary idea is that since u is homeomorphism in $W^{1,2}(\Omega, u(\Omega))$ and satisfies Lusin's condition $\mathcal{N}$, we have, for every measurable $E \subset \Omega$,

$$\int_E J_u = |u(E)| \tag{7.4}$$

(see [MZ92, Theorem 3.1] when E is open and use approximation for general measurable E).

If $E \subset \Omega$ is measurable, then (7.3) follows from (7.4) and the assumption that $J_u > 0$ a.e. in Ω.

In general, if $E \subset \Omega$ and $|u(E)| = 0$, choose a Borel set B such that

$$u(E) \subset B \subset u(\Omega) \quad \text{and} \quad |B| = |u(E)| = 0. \tag{7.5}$$

Since $|B| = 0$ and $u^{-1}(B)$ is measurable, it follows from the case covered above that $|u^{-1}(B)| = 0$. Thus $|E| = 0$. □

Lemma 7.3 implies that u and u^{-1} preserve Lebesgue measurable sets. Thus, in particular, if $g\colon u(\Omega) \to \mathbb{R}$ is a measurable function, then $g \circ u$ is measurable. The following change-of-variables result is a special case of [Haj93, Theorem 2].

Lemma 7.4. *Suppose $\Omega \subset \mathbb{C}$ is open, $u \in W^{1,2}(\Omega, u(\Omega))$ is a homeomorphism and $J_u > 0$ a.e. If $g\colon u(\Omega) \to \mathbb{R}$ is measurable and $g \circ u J_u \in L^1(\Omega)$, then $g \in L^1(u(\Omega))$ and*

$$\int_\Omega g \circ u J_u = \int_{u(\Omega)} g. \tag{7.6}$$

The following lemma is a modification of [CK09, Lemma 3.4].

Lemma 7.5. *Suppose $u \in \dot{W}^{1,2}(\mathbb{C}, \mathbb{C})$ and $b \in \mathrm{BMO}(\mathbb{C})$ satisfy the assumptions of Theorem 7.1. Then*

$$2\mathbf{Re} \int_\Omega u_{\bar z} \overline{(\eta \circ u)_{\bar z}} = \int_\Omega b \operatorname{div} \eta \circ u J_u$$

for every $\eta \in C_0^\infty(u(\Omega), \mathbb{C})$.

Proof. First calculate

$$\begin{aligned} 2\mathbf{Re}\left(u_z \overline{(\eta \circ u)_z} - u_{\bar z} \overline{(\eta \circ u)_{\bar z}}\right) &= 2\mathbf{Re}\left(u_z \overline{\eta_u u_z + \eta_{\bar u} \overline{u_{\bar z}}} - u_{\bar z}\, \overline{\eta_u u_{\bar z} + \eta_{\bar u} \overline{u_z}}\right) \\ &= 2\mathbf{Re}(\overline{\eta_u} J_u) = \operatorname{div} \eta \circ u J_u. \end{aligned}$$

Then note that $\chi_\Omega \cdot \eta \circ u \in W_0^{1,2}(\Omega, \mathbb{C})$ and use condition (7.1) to conclude that

$$2\mathbf{Re}\int_\Omega u_{\bar z}\overline{(\eta\circ u)_{\bar z}} = \int^*_{\mathbb{C}} b\chi_\Omega \operatorname{div}\eta\circ u J_u = \int_\Omega b \operatorname{div}\eta\circ u J_u.$$

□

With the auxiliary results established we are ready to prove Theorem 7.1.

Proof of Theorem 7.1. We want to prove that $b\circ v \in L^1_{loc}(u(\Omega))$ and set $q := b\circ v$. Lemma 7.3 implies that u maps measurable sets onto measurable sets. Hence, $b\circ v$ is measurable.

Our next goal is to prove that $b\circ v \in L^1_{loc}(u(\Omega))$; by Lemma 7.4 it suffices to show that $bJ_u \in L^1_{loc}(\Omega)$. Let $K \subset \Omega$ be compact and choose a square $Q \subset \mathbb{C}$ that contains K. First calculate

$$\int_K |bJ_u| \le \int_K |b - b_Q| J_u + |b_Q| \int_K J_u.$$

Next let $C > 0$ and use the elementary inequality $ab \le e^a - 1 + b\log(1+b)$ for $a, b \ge 0$ to write

$$\int_K |b - b_Q| J_u \le \int_Q \left(\exp\left(\frac{|b-b_Q|}{C}\right) - 1\right) + \int_K CJ_u \log(1 + CJ_u). \tag{7.7}$$

When $C > 0$ is chosen large enough, the John-Nirenberg theorem (see e.g. [Gra04, Corollary 7.1.7]) implies that the first integral on the right hand side of (7.7) converges. Since $u \in W^{1,2}(\Omega, \mathbb{C})$ and $CJ_u > 0$ a.e. in Ω, Theorem 2.17 yields $CJ_u \log(1 + CJ_u) \in L^1_{loc}(\Omega)$, and so the second integral converges as well. Thus $bJ_u \in L^1_{loc}(\Omega)$ and therefore $b\circ v \in L^1_{loc}(u(\Omega))$.

We set $q := b\circ v$ and show that (7.2) holds in the sense of distributions. To this end, let $\eta \in C_0^\infty(u(\Omega), \mathbb{C})$. Since $bJ_u \in L^1_{loc}(\Omega)$, we have $b \operatorname{div}\eta\circ u J_u \in L^1(\Omega)$. We may thus write, using Lemmas 7.4 and 7.5,

$$\begin{aligned} 2\,\mathbf{Re}\int_{u(\Omega)} q\,\eta_\zeta &= \int_{u(\Omega)} q \operatorname{div}\eta = \int_\Omega b \operatorname{div}\eta\circ u J_u = 2\,\mathbf{Re}\int_\Omega u_{\bar z}\overline{(\eta\circ u)_{\bar z}} \\ &= 2\,\mathbf{Re}\int_\Omega |u_{\bar z}|^2 \eta_u + 2\mathbf{Re}\int_\Omega u_z u_{\bar z}\overline{\eta_{\bar u}}. \end{aligned} \tag{7.8}$$

Since $u \in W^{1,2}(\Omega, u(\Omega))$ is a homeomorphism, u is differentiable a.e. (see e.g. [AIM09, Corollary 3.3.3]). Furthermore, by assumption, $J_u > 0$ a.e. in Ω. When u is differentiable and Du is invertible at $z \in \Omega$, we have $u_{\bar z}(z) = -v_{\bar\zeta}(u(z))J_u(z) = -v_{\bar\zeta}(u(z))/J_v(u(z))$ (see [AIM09, p. 34]). By using this identity a.e. in Ω and Lemma 7.4 we get

$$\int_\Omega |u_{\bar z}|^2 = \int_\Omega \frac{|v_{\bar\zeta}\circ u|^2}{J_v\circ u} J_u = \int_{u(\Omega)} \frac{|v_{\bar\zeta}|^2}{J_v},$$

and so $|v_{\bar\zeta}|^2/J_v \in L^1(u(\Omega))$. In a similar vein,

$$2\,\mathbf{Re}\int_\Omega |u_{\bar z}|^2\eta_u = 2\,\mathbf{Re}\int_\Omega \frac{|v_{\bar\zeta}\circ u|^2}{J_v\circ u}\eta_\zeta\circ u J_u = 2\,\mathbf{Re}\int_{u(\Omega)} \frac{|v_{\bar\zeta}|^2}{J_v}\eta_\zeta. \tag{7.9}$$

Likewise, by using the identity $u_z(z) = \overline{v_\zeta(u(z))J_u(z)} = \overline{v_\zeta(u(z))}/J_v(u(z))$ a.e. $z \in \Omega$ and Lemma 7.4 we get $v_\zeta \overline{v_{\bar\zeta}}/J_v \in L^1(u(\Omega), \mathbb{C})$ and

$$(7.10) \qquad 2\,\mathbf{Re}\int_\Omega u_z u_{\bar z}\overline{\eta_{\bar u}} = -2\,\mathbf{Re}\int_{u(\Omega)} \frac{\overline{v_\zeta} v_{\bar\zeta}}{J_v}\,\overline{\eta_{\bar\zeta}} = -2\,\mathbf{Re}\int_{u(\Omega)} \frac{v_\zeta \overline{v_{\bar\zeta}}}{J_v}\,\eta_{\bar\zeta}.$$

By combining (7.8), (7.9) and (7.10) we obtain (7.2), and so the proof of the theorem is complete. □

The local representation of Lagrange multipliers given by Theorem 7.1 allows us to draw the following more global conclusion on the uniqueness of Lagrange multipliers.

Corollary 7.6. *Let $b \in \mathrm{BMO}(\mathbb{C})$ and $u \in \dot W^{1,2}(\mathbb{C}, \mathbb{C})$. Suppose $\Omega \subset \mathbb{C}$ is a domain such that in Ω, u satisfies $J_u > 0$ a.e. and has integrable distortion. If b and u satisfy*

$$(7.11) \qquad \int_{\mathbb{C}}^{*} b\mathbf{Re}(u_z\overline{\phi_z} - u_{\bar z}\overline{\phi_{\bar z}}) = \mathbf{Re}\int_{\mathbb{C}} u_{\bar z}\overline{\phi_{\bar z}} \quad \textit{for all } \phi \in W_0^{1,2}(\Omega, \mathbb{C}),$$

then $b|_\Omega$ is unique up to an additive constant.

Proof. By Corollary 2.8, $u|_\Omega$ is a local homeomorphism outside a discrete set S. When $z \in \Omega \setminus S$, choose a smooth neighborhood $U_z \ni z$ such that $u\colon U_z \to u(U_z)$ is a homeomorphism.

Suppose $b_1, b_2 \in \mathrm{BMO}(\mathbb{C})$ both satisfy (7.11). By Theorem 7.1, there exist $q_1, q_2 \in L^1_{loc}(u(U_z))$ such that $b_1 = q_1 \circ u$ and $b_2 = q_2 \circ u$ in U_z. Since q_1 and q_2 are real-valued, (7.2) implies that $\nabla(q_1 - q_2) = 0$. Hence, $b_1 - b_2$ is constant in U_z.

We still need to make sure that $b_1 - b_2$ equals the same constant in every neighborhood U_z. Suppose $K \subset \Omega \setminus S$ is compact. Then $\{U_z\colon z \in K\}$ is a cover of K, and by the compactness of K it has a finite subcover. It is now a standard topological fact that b is constant in K. It follows that $b_1 - b_2$ is constant in the whole domain Ω. □

The interesting feature of Corollary 7.6 is that we do not need any assumptions about the behavior of b outside Ω in order to reach the conclusion that $b|_\Omega$ is unique. This makes it appear likely that if an energy minimizer $u \in \dot W^{1,2}(\mathbb{C}, \mathbb{C})$ has a Lagrange multiplier $b \in \mathrm{BMO}(\mathbb{C})$, then b is unique.

7.2. **Smooth, compactly supported Lagrange multipliers.** The Euler-Lagrange equations presented in Proposition 4.8 are somewhat difficult to study because various algebraic manipulations are difficult (or impossible) to justify. This problem does not, however, exist when the Lagrange multiplier is smooth enough. We therefore investigate the interplay between an energy-minimal solution u and its Lagrange multiplier b in the model case where b is smooth and compactly supported. The assumption $b \in C_0^\infty(\mathbb{C})$ is natural since $C_0^\infty(\mathbb{C})$ is dense in $\mathrm{VMO}(\mathbb{C})$.

Theorem 7.7. *Suppose $b \in C_0^\infty(\mathbb{C})$ is a Lagrange multiplier for $u \in \dot W^{1,2}(\mathbb{C}, \mathbb{C}) \setminus \{0\}$. Then $u \in C^\infty(\mathbb{C}, \mathbb{C})$.*

Proof. We shall prove that $\mathrm{D}u \in \cup_{n=0}^\infty W^{n,2}(\mathbb{C}, \mathbb{R}^{2\times 2})$. Then the smoothness of u follows from the Sobolev embedding theorem; $u \in \cup_{n=1}^\infty W^{n,2}_{loc}(\mathbb{C}, \mathbb{C}) \subset C^\infty(\mathbb{C}, \mathbb{C})$ (see e.g. [AF03, Theorem 4.12]).

First note that $\mathrm{D}u \in L^2(\mathbb{C}, \mathbb{R}^{2\times 2})$ since $u \in \dot W^{1,2}(\mathbb{C}, \mathbb{C})$. We make the inductive assumption that $\mathrm{D}u \in W^{n,2}(\mathbb{C}, \mathbb{R}^{2\times 2})$ for $n \geq 0$ and intend to show that $\mathrm{D}u \in W^{n+1,2}(\mathbb{C}, \mathbb{R}^{2\times 2})$.

Condition (4.8) in Proposition 4.8 and the assumption $b \in C_0^\infty(\mathbb{C})$ imply that

$$\Delta u_1 = 2(\partial_x b\, \partial_y u_2 - \partial_y b\, \partial_x u_2) \in W^{n,2}(\mathbb{C}). \tag{7.12}$$

Fix a multi-index $\alpha = (\alpha_1, \alpha_2)$ with $|\alpha| = \alpha_1 + \alpha_2 = n$. Then $\partial^\alpha \Delta u_1 \in L^2(\mathbb{C}, \mathbb{C})$ by (7.12). Therefore, by using the properties of second-order Riesz transforms $\mathbf{R}_{ij}$ (see [Gra04]) and in particular the boundedness of $\mathbf{R}_{ij}$ in $L^2(\mathbb{C}, \mathbb{C})$ we may write

$$\partial_x \partial_x \partial^\alpha u_1 = -\mathbf{R}_{11} \Delta \partial^\alpha u_1 = -\mathbf{R}_{11} \partial^\alpha \Delta u_1 \in L^2(\mathbb{C}).$$

Using a similar identity on the other second-order partial derivatives we write the Hessian matrix of $\partial^\alpha u_1$ as

$$\mathcal{H}\partial^\alpha u_1 = \begin{bmatrix} \partial_x\partial_x\partial^\alpha u_1 & \partial_x\partial_y\partial^\alpha u_1 \\ \partial_y\partial_x\partial^\alpha u_1 & \partial_y\partial_y\partial^\alpha u_1 \end{bmatrix} = -\begin{bmatrix} \mathbf{R}_{11} & \mathbf{R}_{12} \\ \mathbf{R}_{21} & \mathbf{R}_{22} \end{bmatrix} \circ \Delta \partial^\alpha u_1 \in L^2(\mathbb{C}, \mathbb{R}^{2\times 2}).$$

Thus $\partial^\alpha \mathcal{H} u_1 \in L^2(\mathbb{C}, \mathbb{C})$, and consequently $\mathrm{D}u_1 \in W^{n+1,2}(\mathbb{C}, \mathbb{C})$. Similarly, $\mathrm{D}u_2 \in W^{n+1,2}(\mathbb{C})$. This completes the proof by induction. □

Theorem 7.7 leaves open the question whether the energy-minimal solution $u \in C^\infty(\mathbb{C}, \mathbb{C})$ or the Jacobian J_u can be chosen to be compactly supported. We gain some information on this matter in Corollary 8.10: if $b \in \mathrm{BMO}(\mathbb{C})$ is a Lagrange multiplier for $u \in \dot{W}^{1,2}(\mathbb{C}, \mathbb{C})$ and vanishes in a domain Ω, then either $J_u = 0$ in Ω or $J_u \neq 0$ in a dense open subset of Ω (for the smooth representative of u).

Now that we know that u is smooth if $b \in C_0^\infty(\mathbb{C})$ it is much easier to study the relations between b and u. In the next two results we prove some identities b and u satisfy.

Proposition 7.8. *Suppose $b \in C_0^\infty(\mathbb{C})$ is a Lagrange multiplier for $u \in \dot{W}^{1,2}(\mathbb{C}, \mathbb{C}) \cap C^\infty(\mathbb{C}, \mathbb{C})$. Then b and u satisfy the equation*

$$(u_z \overline{u_{\bar z}})_{\bar z} = J_u b_z. \tag{7.13}$$

Proof. By Proposition 4.8, b and u satisfy

$$u_{z\bar z} = (b u_z)_{\bar z} - (b u_{\bar z})_z = b_{\bar z} u_z - b_z u_{\bar z}.$$

Using that identity we calculate

$$(u_z \overline{u_{\bar z}})_{\bar z} = u_z \overline{u_{z\bar z}} + u_{z\bar z} \overline{u_{\bar z}} = u_z (b_z \overline{u_z} - b_{\bar z} \overline{u_{\bar z}}) + (b_{\bar z} u_z - b_z u_{\bar z}) \overline{u_{\bar z}} = J_u b_z.$$

□

By using the identities $b_x = b_z + b_{\bar z}$ and $b_y = i(b_z - b_{\bar z})$ we may write (7.13) in real notation in the form

$$\Delta u_1 \nabla u_1 + \Delta u_2 \nabla u_2 = 2 J_u \nabla b.$$

The following result gives a slightly more informative version of Theorem 7.1 for Lagrange multipliers in $C_0^\infty(\mathbb{C})$.

Proposition 7.9. *Suppose $b \in C_0^\infty(\mathbb{C})$ is a Lagrange multiplier for $u \in \dot{W}^{1,2}(\mathbb{C}, \mathbb{C}) \cap C^\infty(\mathbb{C}, \mathbb{C})$. If $J_u > 0$ in a domain Ω and $u\colon \Omega \to u(\Omega)$ is a homeomorphism, then b is of the form*

$$b = q \circ u$$

in Ω, where

$$u_{z\bar z} = q_{\bar u} J_u. \tag{7.14}$$

Proof. By Theorem 7.1, b is of the form $b = q \circ u$. Note that $q = b \circ v \in C^\infty(u(\Omega))$. We calculate

$$b_z = q_u u_z + q_{\bar{u}} \overline{u_{\bar{z}}},$$
$$b_{\bar{z}} = q_u u_{\bar{z}} + q_{\bar{u}} \overline{u_z},$$

and by using Proposition 4.8 and the identities above we get

$$u_{z\bar{z}} = b_{\bar{z}} u_z - b_z u_{\bar{z}} = q_{\bar{u}}(|u_z|^2 - |u_{\bar{z}}|^2) = q_{\bar{u}} J_u.$$

□

8. Behavior of Solutions in Domains Where the Jacobian Vanishes

Sobolev mappings behave in an entirely different manner in domains where the Jacobian is positive and ones where the Jacobian vanishes. As an example of this, if an energy-minimal solution $u \in \dot{W}^{1,2}(\mathbb{C},\mathbb{C})$ satisfies $J_u = 0$ in a domain Ω, then $|u(\Omega)| = 0$ (see Proposition 8.4); this should be contrasted with Corollary 2.8. In this chapter we use the calculus of variations to study energy-minimal solutions in domains where the Jacobian vanishes.

A natural first question that arises is whether every energy minimizer must satisfy $\mathrm{D}u = 0$ a.e. in sets where $J_u = 0$. The answer is no, and examples are provided, for instance, in Corollary 8.9. However, in §8.1 we recall a result from [IO13] that implies that if an energy-minimal solution has vanishing Jacobian in a domain Ω, then it satisfies the Hopf-Laplace equation $(u_z \overline{u_{\bar{z}}})_{\bar{z}} = 0$ in Ω. As a consequence of the equations $J_u = 0$ and $(u_z \overline{u_{\bar{z}}})_{\bar{z}} = 0$, either $\mathrm{D}u = 0$ a.e. or $\mathrm{D}u \neq 0$ a.e. in Ω (see Proposition 8.4).

Harmonic mappings are natural solutions of the Hopf-Laplace equation, and in §8.2 we discuss conditions under which an energy-minimal solution is harmonic in a domain where the Jacobian vanishes. We relate the study of harmonicity to Lagrange multipliers in Corollary 8.10.

We also discuss Jacobians in bounded Lipschitz domains of $\mathbb{C}$ and provide a negative answer to Question 1.29 of Z.J. Lou, S.Z. Yang and D.J. Song mentioned in the Introduction.

8.1. **The Hopf-Laplace equation.** In the previous four chapters we have considered energy-minimal solutions mostly via their relation to Lagrange multipliers. In this chapter we study energy-minimizers and other solutions without assuming the existence of a Lagrange multiplier. We get started by presenting a basic variational lemma that follows directly from the definition of an energy-minimal solution.

Lemma 8.1. *Let $u \in \dot{W}^{1,2}(\mathbb{C},\mathbb{C})$ be an energy-minimal solution. Let $\epsilon_0 > 0$ and for $\epsilon \in (-\epsilon_0, \epsilon_0)$ suppose that the mappings $u^\epsilon \in \dot{W}^{1,2}(\mathbb{C},\mathbb{C})$, $-\epsilon_0 < \epsilon < \epsilon_0$, satisfy*

$$u^0 = u \qquad \text{and} \qquad J_{u^\epsilon} = J_u \quad \text{for every } \epsilon \in (-\epsilon_0, \epsilon_0). \tag{8.1}$$

If the derivative

$$\frac{d}{d\epsilon} \int_{\mathbb{R}^2} |u^\epsilon_{\bar{z}}|^2 \bigg|_{\epsilon=0} \tag{8.2}$$

exists, its value is 0.

The mappings u^ϵ treated in Lemma 8.1 are called *variations*. The idea behind Lemma 8.1 is that in some cases one may write the derivative in (8.2) in another form that depends on u and thereby get a partial differential equation that u satisfies.

It is, in general, difficult to find variations that satisfy (8.1); this is due to the nonlinear nature of the Jacobian operator. However, as an example of the applications of Lemma 8.1 we next use so-called inner variations to study energy-minimal solutions in domains where their Jacobian vanishes.

Suppose $J_u = 0$ in an open set $\Omega \subset \mathbb{C}$. When $\eta \in C_0^\infty(\Omega,\mathbb{C})$ and $\epsilon \in \mathbb{R}$ is small enough,

$$J_{u\circ(\mathrm{id}+\epsilon\eta)} = J_u \circ (\mathrm{id}+\epsilon\eta) J_{\mathrm{id}+\epsilon\eta} = 0 \quad \text{in } \Omega.$$

Lemma 8.1 thus applies to the functions defined by $u^\epsilon := u \circ (\mathrm{id} + \epsilon\eta)$ when $\epsilon \in \mathbb{R}$ is small. The derivative in (8.2) is given another form in the following result; for a proof see e.g. [IO13, Lemma 3.1].

Lemma 8.2. *Let $\Omega \subset \mathbb{C}$ be an open set. Suppose $\eta \in C_0^\infty(\Omega, \mathbb{C})$ and an energy minimizer $u \in \dot{W}^{1,2}(\mathbb{C}, \mathbb{C})$ satisfies $J_u = 0$ in Ω. Then*

$$\frac{d}{d\epsilon} \int_{\mathbb{C}} |(u \circ (z + \epsilon\eta))_{\bar{z}}|^2 \bigg|_{\epsilon=0} = 2\mathbf{Re} \int_\Omega u_z \overline{u_{\bar{z}}} \eta_{\bar{z}} \, dz = 0.$$

Lemma 8.2 and Weyl's lemma imply that the integrable function $u_z\overline{u_{\bar{z}}}$ has an analytic representative in Ω. We record this fact in the following result; for more on the Hopf-Laplace equation see [CIKO], [IKO13], [IO13] and the references contained therein.

Theorem 8.3. *Suppose an energy minimizer $u \in \dot{W}^{1,2}(\mathbb{C}, \mathbb{C})$ satisfies $J_u = 0$ in an open set $\Omega \subset \mathbb{C}$. Then u satisfies the Hopf-Laplace equation*

$$(u_z\overline{u_{\bar{z}}})_{\bar{z}} = 0$$

in Ω.

When an energy minimizer satisfies $J_u = 0$ and the Hopf-Laplace equation $(u_z\overline{u_{\bar{z}}})_{\bar{z}} = 0$ in a domain $\Omega \subset \mathbb{C}$, the mapping u satisfies some higher regularity properties in Ω, as the following basic result shows.

Proposition 8.4. *Suppose $\Omega \subset \mathbb{C}$ is a domain and $u \in W^{1,2}(\Omega, \mathbb{C})$ satisfies*

$$J_u = 0 \quad \text{and} \quad (u_z\overline{u_{\bar{z}}})_{\bar{z}} = 0$$

in the sense of distributions. Then u has the following properties:
(i) either $\mathrm{D}\, u(z) = 0$ a.e. or $\mathrm{D}\, u(z) \neq 0$ a.e.
(ii) $u \in W^{1,\infty}_{loc}(\Omega, \mathbb{C})$ but not necessarily $u \in C^1(\Omega, \mathbb{C})$.
(iii) $|u(\Omega)| = 0$.

Proof. We first prove (i). Suppose $\mathrm{D}u = 0$ in a set of positive measure. From $J_u = |u_z|^2 - |u_{\bar{z}}|^2 = 0$ it follows that $|u_z| = |u_{\bar{z}}|$, and that in turn implies the chain of equivalences

$$u_z\overline{u_{\bar{z}}} = 0 \Leftrightarrow u_z = u_{\bar{z}} = 0 \Leftrightarrow \mathrm{D}u = 0. \tag{8.3}$$

Thus $u_z\overline{u_{\bar{z}}}$ vanishes in a set of positive measure in Ω. Since $u_z\overline{u_{\bar{z}}}$ is, by assumption, analytic, it vanishes in the whole domain Ω. By (8.3), $\mathrm{D}u = 0$ a.e. in Ω.

For the proof of (ii) note that since $u_z\overline{u_{\bar{z}}}$ is analytic,

$$|\mathrm{D}u|^2 = 2(|u_z|^2 + |u_{\bar{z}}|^2) = 4|u_z\overline{u_{\bar{z}}}| \in L^\infty_{loc}(\Omega).$$

Thus $u \in W^{1,\infty}_{loc}(\Omega, \mathbb{C})$.

In order to show that u need not belong to $C^1(\Omega, \mathbb{C})$ consider

$$u(z) = u(x + iy) := |x| = \begin{cases} x, & x > 0, \\ -x, & x < 0 \end{cases}$$

in $B(0, 1)$. It satisfies

$$u_z(z) = u_{\bar{z}}(z) = \begin{cases} \frac{1}{2}, & x > 0, \\ -\frac{1}{2}, & x < 0. \end{cases}$$

Hence, $J_u = 0$ and $(u_z\overline{u_{\bar z}})_{\bar z} = 0$ in the sense of distributions. However, $u \notin C^1(B(0,1),\mathbb{C})$.

Let $A \subset \Omega$ be compact. We intend to show that $|u(A)| = 0$; claim (iii) then follows easily. Since $u \in W^{1,\infty}_{loc}(\Omega,\mathbb{C})$, the mapping u is locally Lipschitz in Ω (see [EG92, p. 131]), and so the Area formula (see [EG92, p. 96]) gives

$$\int_{\mathbb{C}} \sharp(A \cap u^{-1}(\{\zeta\}))\, d\zeta = \int_A J_u = 0.$$

Consequently, $u^{-1}(\{\zeta\}) \cap A = \emptyset$ for a.e. $\zeta \in \mathbb{C}$, that is, $|u(A)| = 0$. □

8.2. **Harmonicity of solutions.** Harmonic functions are natural solutions of the Hopf-Laplace equation; if u is harmonic, then

$$(u_z\overline{u_{\bar z}})_{\bar z} = u_z\overline{u_{z\bar z}} + u_{z\bar z}\overline{u_{\bar z}} = 0.$$

Moreover, by a theorem of T. Iwaniec, L. Kovalev and J. Onninen, every continuous, discrete and open mapping in $W^{1,2}_{loc}(\Omega,\mathbb{C})$ that satisfies the Hopf-Laplace equation is harmonic (see [IKO13, Theorem 1.3]).

Remark 8.5. The conditions $J_u = 0$ and $(u_z\overline{u_{\bar z}})_{\bar z} = 0$ do not imply harmonicity of u even when u is smooth. Counterexamples are furnished, for instance, by the mappings defined by $u(z) := e^{i(z+\bar z)}$ in the plane and $u(z) := z/|z|$ in a domain that does not contain the origin.

These considerations motivate the following problem.

Question 8.6. *If $u \in \dot W^{1,2}(\mathbb{C},\mathbb{C})$ and $J_u = 0$ in a domain Ω, under what further conditions on u or J_u is u harmonic in Ω?*

The following theorem provides a partial answer to Question 8.6.

Theorem 8.7. *Suppose $h \in \mathcal{H}^1(\mathbb{C})$ and $h = 0$ in a bounded domain Ω. The following conditions are equivalent:*

(i) There exists $u \in \dot W^{1,2}(\mathbb{C},\mathbb{C})$ such that $J_u = h$ in $\mathbb{C}$ and u is harmonic in Ω.

(ii) There exists $v \in \dot W^{1,2}(\mathbb{C},\mathbb{C})$ such that $J_v = h$ in $\mathbb{C}$ and v is real-valued in Ω.

(iii) There exists $w \in \dot W^{1,2}(\mathbb{C},\mathbb{C})$ such that $J_w = h$ in $\mathbb{C}$ and w is both harmonic and real-valued in Ω.

We will prove implications $(i) \implies (ii)$ and $(ii) \implies (iii)$, the direction $(iii) \implies (i)$ being obvious.

Proof of $(i) \implies (ii)$. Assume u is non-constant in Ω. Then $u_z \not\equiv 0$ in Ω. Since u_z is analytic in Ω, the set

$$S := \{z \in \Omega\colon u_z(z) = 0\} = \{z \in \Omega\colon \overline{u_{\bar z}(z)} = 0\}$$

is discrete. In $\Omega \setminus S$ we have

$$\left(\frac{u_z}{\overline{u_{\bar z}}}\right)_{\bar z} = \frac{\overline{u_{\bar z}}\, u_{z\bar z} - u_z\, \overline{u_{z\bar z}}}{(\overline{u_{\bar z}})^2} = 0 \quad \text{and} \quad \left|\frac{u_z}{\overline{u_{\bar z}}}\right| = 1.$$

By the Maximum modulus principle, $u_z/\overline{u_{\bar z}}$ is constant in $\Omega \setminus S$. Thus there exists $\alpha \in \mathbb{C}$ such that $|\alpha| = 1$ and

$$\overline{\alpha u_{\bar z}} = \alpha u_z$$

in $\Omega \setminus S$. Furthermore, obviously $\overline{\alpha u_{\bar z}} = 0 = \alpha u_z$ in S.

Now αu satisfies

$$(\alpha u)_z = \overline{\alpha u}_z \tag{8.4}$$

and, using (8.4),

$$(\alpha u)_{\bar{z}} = \overline{\overline{\alpha u}_z} = \overline{(\alpha u)_z} = \overline{\alpha u}_{\bar{z}}. \tag{8.5}$$

By combining (8.4) and (8.5) we get $\mathrm{D}(\alpha u - \overline{\alpha u}) = 0$, and so $\mathbf{Im}(\alpha u) =: c \in \mathbb{R}$ is constant in Ω.

Define $v := \alpha u - ic$. Then v is real-valued in Ω and

$$J_v = |(\alpha u - ic)_z|^2 - |(\alpha u - ic)_{\bar{z}}|^2 = J_u = h$$

in the whole plane. Furthermore, since u is an energy-minimal solution, v is an energy-minimal solution as well. ☐

Proof of $(ii) \implies (iii)$. Since Ω is a bounded domain, a Poincaré inequality holds for $W_0^{1,2}(\Omega)$ (see e.g. [Eva98, p. 265]), and by exploiting that we can solve the generalized Dirichlet problem and find $f \in W^{1,2}(\Omega)$ satisfying

$$f_{z\bar{z}} = 0, \quad f - v \in W_0^{1,2}(\Omega) \tag{8.6}$$

(see e.g. [GT01, p. 181]). Define

$$w(z) := \begin{cases} f(z), & z \in \Omega, \\ v(z), & z \in \mathbb{C} \setminus \Omega \end{cases}$$

and note that since zero extensions of mappings in $W_0^{1,2}(\Omega)$ belong to $\dot{W}^{1,2}(\mathbb{C}, \mathbb{C})$, we have $w = v + (f - v)\chi_\Omega \in \dot{W}^{1,2}(\mathbb{C}, \mathbb{C})$. Now $J_w = J_v = h$ and w is harmonic and real-valued in Ω. ☐

The following consequence of Theorem 8.7 gives a partial answer to Question 8.6 in terms of the behavior of the Jacobian.

Corollary 8.8. *Suppose $\Omega \subset \mathbb{C}$ is a domain and Ω' is a smooth, relatively compact subdomain of Ω. If $u \in \dot{W}^{1,2}(\mathbb{C}, \mathbb{C})$ satisfies $\int_{\Omega'} J_u > 0$ and $J_u = 0$ in $\Omega \setminus \Omega'$, then u is not harmonic in $\Omega \setminus \Omega'$.*

Proof. Seeking contradiction, suppose $u \in \dot{W}^{1,2}(\mathbb{C}, \mathbb{C})$ satisfies the assumptions of the corollary and is harmonic in $\Omega \setminus \Omega'$. By Theorem 8.7, we may assume that u is real-valued in $\Omega \setminus \Omega'$. Thus $u - u_1 \in W_0^{1,2}(\Omega, \mathbb{C})$, and so Theorem 2.21 yields

$$\int_{\Omega'} J_u = \int_\Omega J_u = \int_\Omega J_{u_1} = 0.$$

This contradicts the assumption $\int_{\Omega'} J_u > 0$. ☐

Corollary 8.8 is not vacuously true; in fact, the assumptions of Corollary 8.8 hold for any mapping $u \in \dot{W}^{1,2}(\mathbb{C}, \mathbb{C})$ that satisfies $u(z) = z$ when $|z| < 1$ and $u(z) = z/|z|$ when $1 < |z| < 2$. By taking $h = J_u \in \mathcal{H}^1(\mathbb{C})$ for such a mapping u and using Corollary 8.8 we get the following result.

Corollary 8.9. *There exists $h \in \mathcal{H}^1(\mathbb{C})$ such that the equation $J_u = h$ has a solution but no solution is harmonic in the (non-empty) interior of the whole set $\{z \in \mathbb{C} \colon h(z) = 0\}$.*

Theorem 8.7 can also be used to study energy minimizers that possess a Lagrange multiplier.

Corollary 8.10. *Suppose $b \in \mathrm{BMO}(\mathbb{C})$ is a Lagrange multiplier for $u \in \dot{W}^{1,2}(\mathbb{C},\mathbb{C})$. If $b = 0$ in a domain Ω, then either $J_u = 0$ in Ω or $J_u \neq 0$ in an open dense subset of Ω (for the smooth representative of u in Ω).*

Proof. Since $b = 0$ in Ω, it follows from Proposition 4.8 that $u = u_1 + iu_2$ is harmonic in Ω. Suppose the open set $\{z \in \Omega \colon J_u(z) \neq 0\}$ is not dense in Ω. Then $J_u = 0$ in a nonempty open set $\Omega' \subset \Omega$. By Theorem 8.7, we may assume that $u_2 = 0$ in Ω'. Now u_2 is harmonic in Ω and vanishes in the nonempty open set Ω', and therefore $u_2 = 0$ in the whole domain Ω. Consequently, $J_u = 0$ in Ω. □

8.3. **Solution of a problem of Lou, Yang and Song.** One way to approach Question 8.6 is to assume $J_u = 0$ outside a bounded Lipschitz domain Ω and study the way the behavior of J_u inside Ω affects the behavior of u outside Ω. Hardy space theory of Jacobians of Sobolev mappings has been studied in bounded Lipschitz domains of $\mathbb{R}^n$ in, e.g., [CDS99], [Lou05], [CDY10], [HLMZ00] and [LYS05]. Recall from §2.3-2.4 that when Ω is a bounded Lipschitz domain, the Jacobian operator maps the Sobolev space $W_0^{1,2}(\Omega,\mathbb{C})$ into the Hardy-type space $\mathcal{H}^1_z(\Omega)$.

An analogue of the Jacobian decomposition of $\mathcal{H}^1$ functions, Theorem 2.20, can be proved in bounded Lipschitz domains of $\mathbb{C}$. The difference to Theorem 2.20 is that the mappings whose Jacobians form the decomposition belong to $W_0^{1,2}(\Omega,\mathbb{C})$. The proof presented in [LYS05] is slightly erroneous, but it can be corrected (see [Lou05, Proposition 3.2] or [CDS99, pp. 63-64]). Thus, in analogy to the case of the whole plane, no proper closed subspace of $\mathcal{H}^1_z(\Omega)$ contains the Jacobian of every mapping in $W_0^{1,2}(\Omega,\mathbb{C})$. The following problem was posed by Z.J. Lou, S.Z. Yang and D.J. Song in [LYS05].

Question 8.11. *Does the Jacobian operator J map $W_0^{1,2}(\Omega,\mathbb{C})$ onto $\mathcal{H}^1_z(\Omega)$?*

The answer to Question 8.11 is *no*, as the following theorem demonstrates.

Theorem 8.12. *Let $\Omega \subset \mathbb{C}$ be a nonempty bounded Lipschitz domain. If $h \in \mathcal{H}^1_z(\Omega)$ and there exist $z_0 \in \partial\Omega$ and $r > 0$ such that $h \geq C > 0$ in $B(z_0,r) \cap \Omega$, then the equation $J_u = h$ has no solution $u \in W_0^{1,2}(\Omega,\mathbb{C})$.*

Proof. Let $h \in \mathcal{H}^1_z(\Omega)$ satisfy the assumptions of the theorem. Suppose, by way of contradiction, that $u \in W_0^{1,2}(\Omega,\mathbb{C})$ satisfies $J_u = h$. Extend u as zero to $\mathbb{C} \setminus \Omega$. Then u has integrable distortion (see §2.2) in $B(z_0,r)$ as the following calculation shows:

$$\begin{aligned}\int_{B(z_0,r)} K_u &:= \int_{B(z_0,r)\cap\Omega} \frac{\|\mathrm{D}u\|^2}{J_u} + \int_{B(z_0,r)\setminus\Omega} 1 \\ &\leq \int_{B(z_0,r)\cap\Omega} \frac{\|\mathrm{D}u\|^2}{C} + |B(z_0,r)| < \infty.\end{aligned}$$

By Theorem 2.7, $u|_{B(z_0,r)}$ is either constant or an open mapping. However, u is not constant in $B(z_0,r) \cap \Omega$ and not open in $B(z_0,r) \setminus \bar{\Omega}$. Since $z_0 \in \partial\Omega$, it follows easily from the definition of a Lipschitz domain (see §2.1) that the open sets $B(z_0,r)\cap\Omega$ and $B(z_0,r) \setminus \bar{\Omega}$ are nonempty. We get a contradiction, and so a solution $u \in W_0^{1,2}(\Omega,\mathbb{C})$ to the equation $J_u = h$ cannot exist. □

Functions of the kind treated in Theorem 8.12 can be constructed in any nonempty bounded Lipschitz domain. Indeed, take any point $z_0 \in \partial\Omega$, choose $r > 0$ such that $\Omega \setminus B(z_0, r)$ is nonempty and define $h\colon \Omega \to \mathbb{R}$ by

$$h := \chi_{\Omega \cap B(z_0,r)} - \frac{|\Omega \cap B(z_0,r)|}{|\Omega \setminus B(z_0,r)|}\chi_{\Omega \setminus B(z_0,r)}.$$

Then h is bounded and has integral zero, and therefore $h \in \mathcal{H}^1_z(\Omega)$.

Remark 8.13. Even if $h \in \mathcal{H}^1_z(\Omega)$ satisfies the assumptions of Theorem 8.12, there may exist $u \in \dot{W}^{1,2}(\mathbb{C}, \mathbb{C})$ such that $J_u = h$ in Ω and $J_u = 0$ outside Ω. Indeed, in Figure 1 we give the values of a Lipschitz mapping $u \in W^{1,2}_0((-2, 2) \times (-1, 3))$ with Jacobian $J_u = \chi_{(0,1)\times(0,1)} - \chi_{(-1,0)\times(0,1)}$. Now set $\Omega = (-1, 1) \times (0, 1)$ so that $J_u \in \mathcal{H}^1_z(\Omega)$ and $J_u \equiv 1$ in, say, $B(1 + i, 1/2) \cap \Omega$.

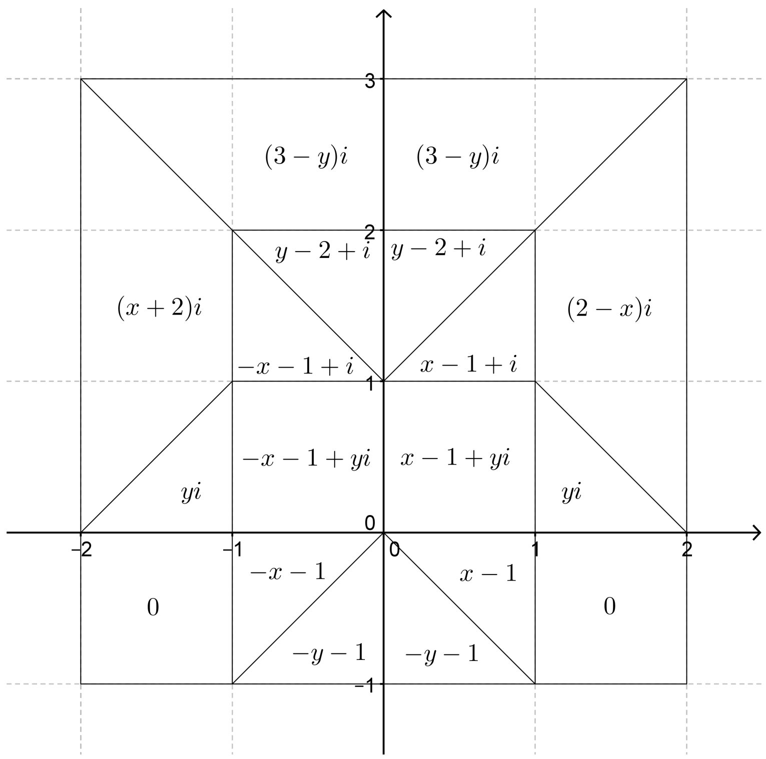

FIGURE 1. Values of the mapping u in $(-2, 2) \times (-1, 3)$

The essential idea of Theorem 8.12 is that if a mapping $u \in \dot{W}^{1,2}(\mathbb{C}, \mathbb{C})$ vanishes on one side of a Lipschitz boundary, then its Jacobian cannot be bounded from below by a positive constant on another, even locally. There is, however, no such restraint on the behavior of the Jacobian if only one of the components of u vanishes outside the domain.

Suppose $\Omega \subset \mathbb{C}$ is a bounded Lipschitz domain and $u \in W^{1,2}(\Omega) + iW^{1,2}_0(\Omega)$. We may choose $U \in W^{1,2}(\mathbb{C}, \mathbb{C})$ such that $U|_\Omega = u$ and $U_2 = 0$ outside Ω (see §2.1). Therefore $J_u = J_U|_\Omega \in \mathcal{H}^1_z(\Omega)$. We end this dissertation by posing a modified version of Question 8.11.

Question 8.14. *Does J map $W^{1,2}(\Omega) + i\,W^{1,2}_0(\Omega)$ onto $\mathcal{H}^1_z(\Omega)$?*

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