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TEICHMÜLLER SPACES OF KLEIN SURFACES

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Contents

Introduction

It was realized as late as in 1969 by N.L. Alling and N. Greenleaf ([3]) that the analytic counterpart of a real algebraic curve is a possibly non-orientable surface with boundary and with a dianalytic structure. That observation motivates the study of these surfaces, known as Klein surfaces.

A real algebraic curve C is a complex curve C_c with an involution σ of C_c. This complex curve corresponds to a compact Riemann surface X_c, and σ induces an antianalytic involution σ of X_c. Our Klein surface X corresponding to C is obtained as the quotient $X = X_c/\sigma$. This representation for a Klein surface has been successfully used to transform results about Riemann surfaces into Klein surfaces ([2], [4], etc.).

In this paper we are interested in the problem of moduli of Klein surfaces. Using the representation $X = X_c/\sigma$ one can transform this problem into symmetric Riemann surfaces (see [9] and the references given there). In that way one can study Teichmüller spaces of Klein surfaces. The author believes, however, that it is useful, or at least instructive, to work directly on Klein surfaces.

As regards the analytic definition of the Teichmüller space $T(X)$ of a Klein surface X, we have two possibilities which lead to different results. While these two definitions are analytically equally well motivated it seems to be natural, from the geometric point of view, to define the Teichmüller space $T(C)$ of a real curve C as the subspace of $T(C_c)$ whose points are represented by real curves "homeomorphic" to C. This leads us to consider a generalization of reduced Teichmüller spaces. We will show in the present paper how Bers' methods (see [1], [7], and [11]) can be applied to such Teichmüller spaces. While preparing this research the author has found O. Lehto's lectures ([11]) especially useful.

In Chapter 1 we recall some definitions about Klein surfaces following Alling and Greenleaf ([4]). We also define quasiconformal mappings of Klein surfaces. The novelty here is that a quasiconformal mapping can be, in some sense, sense-reversing.

In Chapters 2 and 3 we recall some well-known theorems about the uniformization of Klein surfaces (see [13]) and, following Bers ([6]), obtain a version of simultaneous uniformization of compact Klein surfaces.

In Chapter 4 we consider liftings of mappings and their homotopies. We obtain a topological result (Lemma 4.3) which allows us to simplify the definition of the reduced Teichmüller space of a surface with boundary (cf. [10]).

In Chapter 5 we define the Teichmüller space $T(Z)$ of a possibly non-orientable compact two-manifold Z. The definition is analogous to those given in [9] and [12]. Then, following Bers ([1], [7], [11]), we embed the space $T(Z)$ into a suitable space of quadratic differentials. Using this embedding we prove that $T(Z)$ is homeomorphic to a simply connected open subdomain of a Euclidean space. If Z is non-orientable or has a non-empty boundary, the real dimension of $T(Z)$ is $3g(Z) - 3$; otherwise it is $6g(Z) - 6$. Here $g(Z)$ denotes the algebraic genus of Z.

In the end we prove, relying on classical results, that $T(Z)$ has a natural real analytic structure such that the above embedding is a real analytic mapping of $T(Z)$ into a real Banach space.

1. Klein surfaces

1.1. In this paper we are dealing with a topological two-manifold Z which may have boundary; the boundary of Z is denoted by ∂Z. We call an atlas $\tilde{U} = \{(U_i, z_i) | i \in I\}$ of Z <u>dianalytic</u> if the coordinate transition functions $z_i \circ z_j^{-1}$ belong to the class C^2 and satisfy either $\partial(z_i \circ z_j^{-1})/\partial\bar{z} = 0$ or $\partial(z_i \circ z_j^{-1})/\partial z = 0$ in a neighborhood of every interior point of $z_j(U_i \cap U_j)$ in the complex plane $\mathbb{C}$.

For notational convenience we shall assume here that each set $z_i(U_i)$ is an open subset of the closed upper half-plane $\bar{H} = \{z \in \mathbb{C} | \operatorname{Im} z \geq 0\}$.

Two dianalytic atlases $\tilde{U}$ and $\tilde{V}$ are called equivalent if $\tilde{U} \cup \tilde{V}$

is a dianalytic atlas of Z; an equivalence class X of dianalytic atlases of Z is called a dianalytic structure of Z. A topological two-manifold Z with a dianalytic structure X is called a Klein surface. The correct notation (Z,X) for a Klein surface is usually abbreviated to X.

Since the coordinate transition functions of a dianalytic atlas can be sense-reversing, a Klein surface need not be orientable. For example, the real projective plane is a Klein surface.

On the other hand, every Riemann surface has a natural dianalytic structure; Klein surfaces arising from Riemann surfaces in this manner are called classical. A Klein surface is non-classical if it has boundary or is non-orientable.

Let $f: X \to Y$ be a continuous mapping of Klein surfaces which maps ∂X into ∂Y. Let $\kappa: \mathbb{C} \to \mathbb{C}$, $x + iy \to x + i|y|$, be the folding map. We call f a morphism of Klein surfaces if for each $p \in X$ and for all local variables z and w about p and $f(p)$, respectively, we can find a holomorphic function F such that $w \circ f = \kappa \circ F \circ z$ holds in some neighborhood of p. A morphism is an isomorphism if it is a homeomorphism.

1.2. We will transform results about Riemann surfaces into Klein surfaces using two kinds of coverings of Klein surfaces with Riemann surfaces.

The orienting double of a Klein surface X is a triple (X_o,π_o,σ_o) satisfying the following conditions:

(i) X_o is a Riemann surface, possibly with boundary, or a disjoint union of two of them.

(ii) The morphism $\pi_o: X_o \to X$ is a two-to-one covering map which is locally a homeomorphism.

(iii) $\sigma_o: X_o \to X_o$ is an antianalytic involution satisfying $\pi_o = \pi_o \circ \sigma_o$.

By an explicit construction we can show that every Klein surface X has an orienting double X_o which will be connected if and only if X is non-orientable. X_o has a non-empty boundary if and only if X has. ∂X_o consists, in fact, of two parts which are mapped onto each other by σ_o and which are mapped homeomorphically onto ∂X by π_o.

The complex double of a Klein surface X is a triple (X_c,π,σ) satisfying the following conditions:

(i) X_c is a Riemann surface or a disjoint union of two of them.
(ii) The morphism $\pi: X_c \to X$ is a double covering map which is two-to-one and locally homeomorphic outside of $\pi^{-1}(\partial X)$ and one-to-one in $\pi^{-1}(\partial X)$.
(iii) $\sigma: X_c \to X_c$ is an antianalytic involution satisfying $\pi = \pi \circ \sigma$.

If $\partial X = \emptyset$, then the orienting double is also the complex double. If $\partial X \neq \emptyset$, then ∂X_o consists of two symmetric parts both homeomorphic to ∂X. Gluing them together we obtain X_c. (π and σ are then the mappings of X_c defined by the mappings π_o and σ_o of X_o.) Hence every Klein surface has also a complex double. One can show that it is unique up to an isomorphism ([4], Proposition 1.6.2).

A Riemann surface admitting antianalytic involutions is called <u>symmetric</u>; an antianalytic involution of a symmetric Riemann surface is called a <u>symmetry</u>. The complex double X_c (the orienting double X_o) of a Klein surface X is a symmetric Riemann surface with symmetry σ (σ_o). From (ii) and (iii) it follows that $X = X_c/\sigma$ ($X = X_o/\sigma_o$).

Conversely, given a symmetric Riemann surface S without boundary, if φ is a symmetry on S, then S/φ is a Klein surface whose complex double is S.

1.3. Consider a collection $f_{\tilde{U}} = \{f_i\}$ of functions $f_i: U_i \to \hat{\mathbb{C}}$, where $\hat{\mathbb{C}}$ denotes the Riemann sphere, defined on charts of the dianalytic atlas $\tilde{U} = \{(U_i, z_i) | i \in I\}$ of a Klein surface X. We recall that $f_{\tilde{U}}$ is a <u>meromorphic function</u> relative to $\tilde{U}$ if the following holds:
(i) Each function $f_i \circ z_i^{-1}$ is meromorphic, and $f_i(\partial X \cap U_i)$ is a subset of the extended real line.
(ii) Let $p \in U_i \cap U_j$. Then if $z_i \circ z_j^{-1}$ is holomorphic at $z_j(p)$, $f_i(p) = f_j(p)$; otherwise $\overline{f_i(p)} = f_j(p)$.

If $f_{\tilde{U}}$ and $f_{\tilde{V}}$ are meromorphic functions relative to $\tilde{U}$ and $\tilde{V}$, respectively, then they are equal if $f_{\tilde{U}} \cup f_{\tilde{V}}$ is a meromorphic function relative to $\tilde{U} \cup \tilde{V}$. An equivalence class $\underline{f}$ of meromorphic functions relative to dianalytic atlases of X is a <u>meromorphic function</u> on X.

Meromorphic functions on X form, in an obvious way, a field $M(X)$. It contains $\mathbb{C}$ as a subfield if and only if X is classical. If X is not classical, then $M(X)$ is a field extension of $\mathbb{R}$, the field of real numbers. If X is compact, the transcendence degree of the field extension $M(X)/\mathbb{R}$ is 1.

Let $\varphi: X \to Y$ be a morphism of Klein surfaces, and let $\tilde{U} = \{(U_i, z_i) | i \in I\}$ and $\tilde{V} = \{(V_j, w_j) | j \in J\}$ be dianalytic atlases of X and Y such that each chart of $\tilde{U}$ is connected and each $\varphi(U_i)$ is contained in some V_j. Consider a meromorphic function $\underline{f} \in M(Y)$ represented by $f_{\tilde{V}} = \{f_j\}$. For the moment, let α denote the mapping of $\hat{\mathbb{C}}$ onto itself which takes a complex number to its complex conjugate and ∞ to ∞.

Choose an $(U_i, z_i) \in \tilde{U}$, and let $(V_j, w_j) \in \tilde{V}$ be such that $\varphi(U_i) \subset V_j$. Since $w_j(V_j)$ is a subset of the closed upper half-plane, we can continue, by the Schwarz reflection principle, the meromorphic function $f_j \circ w_j^{-1}$ to a meromorphic function $\widehat{f_j \circ w_j^{-1}}$ defined in the set $w_j(V_j) \cup \alpha(w_j(V_j))$. Let now Φ be the holomorphic function satisfying $w_j \circ \varphi = \kappa \circ \Phi \circ z_i$ in U_i. (Eventually we can find such an Φ.) Denote $\widehat{f_j \circ w_i^{-1}} \circ \Phi \circ z$ by $\varphi^*(\underline{f})_i$. The family $\varphi^*(\underline{f})_{\tilde{U}} = \{\varphi^*(\underline{f})_i | i \in I\}$ is then a meromorphic function relative to $\tilde{U}$. Let $\varphi^*(\underline{f})$ denote its equivalence class. After all these definitions we obtain a well defined homomorphism $\varphi^*: M(Y) \to M(X)$, $\underline{f} \to \varphi^*(\underline{f})$.

If φ is not constant, φ^* is a monomorphism. If (X_c, π, σ) is the complex double of X, then $\pi^*(M(X))$ equals to the subfield of $M(X_c)$ left fixed by σ^* ([4], Theorem 1.6.4).

1.4. Let $\tilde{U} = \{(U_i, z_i) | i \in I\}$ be a dianalytic atlas of X. If $U_i \cap U_j \neq \emptyset$, we define the function $T_{ij}: U_i \cap U_j \to \mathbb{C}$ by

$$T_{ij} = \left(\frac{\partial}{\partial z}(z_i \circ z_j^{-1}) + \frac{\partial}{\partial \bar{z}}(z_i \circ z_j^{-1})\right) \circ z_j = \left(\frac{\partial}{\partial x}(z_i \circ z_j^{-1})\right) \circ z_j,$$

and recall that a family $\omega_{\tilde{U}} = \{\omega_i | i \in I\}$ of meromorphic functions $\omega_i: U_i \to \hat{\mathbb{C}}$ is a <u>meromorphic differential</u> relative to $\tilde{U}$ if:

(i) $\omega_i(\partial X \cap U_i) \subset \mathbb{R} \cup \{\infty\}$ for each $i \in I$.

(ii) If $z_i \circ z_j^{-1}$ is holomorphic at $z_j(p)$, then $\omega_j(p) = \omega_i(p) T_{ij}(p)$; otherwise $\overline{\omega_j(p)} = \omega_i(p) T_{ij}(p)$.

Meromorphic differentials $\omega_{\tilde{U}}$ and $\omega_{\tilde{V}}$ relative to atlases $\tilde{U}$ and $\tilde{V}$, respectively, are called equal if $\omega_{\tilde{U}} \cup \omega_{\tilde{V}}$ is a meromorphic differential relative to $\tilde{U} \cup \tilde{V}$. Equivalence classes ω of meromorphic differentials relative to dianalytic atlases of X are called <u>meromorphic differentials</u> on X. They form a vector space $D(X)$ which is a complex vector space if X is classical and real otherwise.

If $\underline{f} \in M(X)$, then taking the derivates locally we can define the differential $d\underline{f}$ of $\underline{f}$ which turns out to be a meromorphic differential. If $\underline{f}$ is not constant, then $d\underline{f} \neq 0$.

The product $\underline{f}\omega$ of a meromorphic function $\underline{f}$ and a meromorphic differential ω is locally well-defined. Checking the transformation rule we see that $\underline{f}\omega$ is, in fact, a meromorphic differential on X. And if $\underline{f} \in M(X)$ is not constant, then every $\omega \in D(X)$ can be written in the form

$$(1.1) \qquad \omega = \underline{g}d\underline{f},$$

where $\underline{g} \in M(X)$ is uniquely defined by ω. Hence $D(X)$ is a one-dimensional vector space over $M(X)$.

A non-constant morphism $\phi: X \to Y$ of Klein surfaces induces a monomorphism $\phi^*: M(Y) \to M(X)$. Using (1.1) we can define the mapping $\phi^*: D(Y) \to D(X)$ induced by ϕ setting $\phi^*(\underline{g}d\underline{f}) = \phi^*(\underline{g})d\phi^*(\underline{f})$, where the ϕ^*'s on the right hand side denote the mapping $\phi^*: M(Y) \to M(X)$. The definition does not depend on the choice of $\underline{f}$. It is clear that the mapping ϕ^* is a $M(Y)$-linear injection of $D(Y)$ into $D(X)$.

Let (X_c, π, σ) be the complex double of X. Checking the definitions it is easy to see that $\pi^*(D(X))$ equals to the subset of $D(X_c)$ left fixed by σ^*. (Hence the differentials on X can be characterized as the differentials of X_c which take conjugate "values" at symmetric points of X_c.)

A differential $\omega \in D(X)$ is <u>holomorphic</u> if it is defined by a family of holomorphic functions. Holomorphic differentials form a subspace $D_o(X)$ of $D(X)$ which is finite dimensional if X is compact. (This follows immediately from the corresponding result about Riemann surfaces and the above remark.)

1.5. Consider again the atlas $\tilde{U} = \{(U_i, z_i) | i \in I\}$ of X. We will say that a family $\alpha_{\tilde{U}} = \{\alpha_i | i \in I\}$ of meromorphic functions $\alpha_i: U_i \to \hat{\mathbb{C}}$ is a <u>meromorphic</u> <u>quadratic</u> <u>differential</u> relative to $\tilde{U}$ if the following holds:

(i) $\alpha_i(\partial X \cap U_i) \subset \mathbb{R} \cup \{\infty\}$ for each $i \in I$.

(ii) If $z_i \circ z_j^{-1}$ is holomorphic at $z_j(p)$, then $\alpha_j(p) = \alpha_i(p)T_{ij}(p)^2$; otherwise $\overline{\alpha_j(p)} = \alpha_i(p)T_{ij}(p)^2$ for all $p \in U_i \cap U_j$.

In the set of meromorphic quadratic differentials relative to dianalytic atlases of X we introduce an equivalence relation in the usual way: $\alpha_{\tilde{U}}$ and $\alpha_{\tilde{V}}$ are equivalent if $\alpha_{\tilde{U}} \cup \alpha_{\tilde{V}}$ is a quadratic differential relative to $\tilde{U} \cup \tilde{V}$. Equivalence classes of meromorphic quadratic differentials relative to atlases of X are called <u>meromorphic quadratic differentials</u> on X. They form a vector space $D^2(X)$. If a meromorphic quadratic differential is defined by a family of holomorphic functions, it is called <u>holomorphic</u>. The space $D_o^2(X)$ of holomorphic quadratic differentials on X is a subspace of $D^2(X)$. $D_o^2(X)$ and $D^2(X)$ are complex vector spaces if X is classical, otherwise real. If X is compact, $D_o^2(X)$ is finite dimensional.

Let $\omega_{\tilde{U}} = \{\omega_i | i \in I\} \in \omega$ and $\tau_{\tilde{U}} = \{\tau_i | i \in I\} \in \tau$ be meromorphic (holomorphic) differentials relative to $\tilde{U}$. The family $\omega_{\tilde{U}}\tau_{\tilde{U}} = \{\omega_i\tau_i\}$ is then a meromorphic (holomorphic) quadratic differential relative to $\tilde{U}$; its equivalence class $\omega\tau$ is a meromorphic (holomorphic) quadratic differential on X which we call the product of ω and τ. The definition of $\omega\tau$ does not depend on the choice of the dianalytic atlas $\tilde{U}$.

Conversely, if $\alpha_{\tilde{U}} = \{\alpha_i | i \in I\} \in \alpha$ is a quadratic differential relative to $\tilde{U}$ and if $\omega_{\tilde{U}} = \{\omega_i | i \in I\} \in \omega$ is a non-constant differential relative to $\tilde{U}$, then the family $\{\alpha_i/\omega_i | i \in I\}$ is a meromorphic differential relative to $\tilde{U}$; denote its equivalence class by α/ω. Then it is clear that for all $\alpha \in D^2(X)$ and for all non-zero $\omega \in D(X)$ we can write

$$(1.2) \qquad \alpha = \tau\omega,$$

where $\tau = \alpha/\omega \in D(X)$ is uniquely determined by ω.

Let $\phi: X \to Y$ be a non-constant morphism. In 1.4 we have seen that it defines a linear injection $\phi^*: D(Y) \to D(X)$. Using (1.2) we can define the corresponding mapping $\phi^*: D^2(Y) \to D^2(X)$ setting $\phi^*(\alpha) = \phi^*(\tau)\phi^*(\omega)$, where the ϕ^*'s on the right hand side denote the mapping $\phi^*: D(Y) \to D(X)$. Then ϕ^* is a well-defined linear injection of $D^2(Y)$ into $D^2(X)$ which maps $D_o^2(Y)$ into $D_o^2(X)$.

If (X_c, π, σ) is the complex double of X, then $D^2(X)$ and $D_o^2(X)$ are, as vector spaces, isomorphic to $\pi^*(D^2(X))$ and $\pi^*(D_o^2(X))$, respectively. And it follows from all the given definitions that quadratic differentials on X can be characterized as quadratic differentials on X_c taking conjugate "values" at symmetric points.

1.6. In this section we assume that X is a compact Klein surface. Then the vector space of holomorphic differentials on X, $D_o(X)$, is finite dimensional. We recall that if X is classical, $D_o(X)$ is a $\mathbb{C}$-vector space, being otherwise a $\mathbb{R}$-vector space, and define the <u>algebraic genus</u> of X, $g(X)$, by

$$g(X) = \begin{cases} \dim_{\mathbb{C}} D_o(X) & \text{if } X \text{ is classical} \\ \dim_{\mathbb{R}} D_o(X) & \text{if } X \text{ is non-classical.} \end{cases}$$

L e m m a 1.1. *If* X *is a compact non-classical Klein surface, then* $g(X_c) = g(X)$.

Proof. Let $\tilde{\omega} \in D_o(X_c)$. The differentials $\tilde{\omega}_1 = (\tilde{\omega} + \sigma^*(\tilde{\omega}))/2$ and $\tilde{\omega}_2 = (\tilde{\omega} - \sigma^*(\tilde{\omega}))/2i$ satisfy $\tilde{\omega}_j = \sigma^*(\tilde{\omega}_j)$, $j = 1,2$, and hence define differentials $\omega_j \in D_o(X)$ for which $\tilde{\omega}_j = \pi^*(\omega_j)$, $j = 1,2$. It follows that any $\tilde{\tau} \in D_o(X_c)$ can be written in the form $\tilde{\tau} = \pi^*(\tau_1) + i\pi^*(\tau_2)$, where $\tau_j \in D_o(X)$, $j = 1,2$. If now the differentials $\omega_1, \omega_2, \ldots, \omega_n$ form a basis for the real vector space $D_o(X)$, then the differentials $\pi^*(\omega_1), \pi^*(\omega_2), \ldots, \pi^*(\omega_n)$ form a basis for the complex vector space $D_o(X_c)$. Hence $\dim_{\mathbb{R}} D_o(X) = \dim_{\mathbb{C}} D_o(X_c)$. The lemma is proved.

For compact Klein surfaces X we define the <u>topological genus</u> $p(X)$ as follows. If X is classical, $p(X)$ denotes the number of tori which connected together give X. If X is not orientable and $\partial X = \phi$, then $p(X)$ is the number of real projective planes needed to build X. If $\partial X \neq \phi$, then the Schottky double, X_S, of X, which is obtained by gluing two copies of X together along the boundary, is a Klein surface without boundary. And we define $p(X) = p(X_S)$.

In addition to the topological genus also the Euler characteristic $\chi(X)$ of X is used in classification of compact topological surfaces. For clarity we give the relations between g, p and χ in the following table ([2], Theorem 1.1).

	$\partial X = \phi$ $k(X) = 0$	$\partial X \neq \phi$ $k(X) = 0$	$\partial X = \phi$ $k(X) = 1$	$\partial X \neq \phi$ $k(X) = 1$
$p(X) =$	$g(X)$	$g(X)$	$g(X) + 1$	$2g(X)$
$\chi(X) =$	$2 - 2g(X)$	$1 - g(X)$	$1 - g(X)$	$1 - g(X)$

Here $k(X)$ denotes the index of orientability being 0 if X is orientable and 1 otherwise.

R e m a r k. By definition, the Euler characteristic and the topological genus of a Klein surface depend only on the underlying topological space, while the algebraic genus depends, a priori, also on the dianalytic structure. The above table shows, however, that also $g(X)$ is defined by the topological type of X. Since any compact topological two-manifold carries a dianalytic structure, we can, by this observation, speak of its algebraic genus.

1.7. In the classical case we know that for compact Klein surfaces X $\dim_{\mathbb{C}} D_o^2(X) = 3g(X) - 3$. If X is not classical, then we can employ the reasoning of the proof of Lemma 1.1 once more to obtain the following:

T h e o r e m 1.1. *Let* X *be a compact non-classical Klein surface. Then* $\dim_{\mathbb{R}} D_o^2(X) = 3g(X) - 3$.

1.8. Let $\tilde{U} = \{(U_i, z_i) | i \in I\}$ be again a dianalytic atlas of a Klein surface X. Let the functions T_{ij} be defined as in 1.4. We consider a family $\mu_{\tilde{U}} = \{\mu_i | i \in I\}$ of functions $\mu_i\colon U_i \to \mathbb{C}$ subject to the following conditions:

(1.3)

(i) Each function $\mu_i \circ z_i^{-1}$ is measurable with respect to the Lebesgue measure in $\mathbb{C}$.

(ii) Assume that $U_i \cap U_j \neq \phi$. If $z_i \circ z_j^{-1}$ is holomorphic at $z_j(p)$, then $\mu_j(p) = \mu_i(p)\overline{T_{ij}(p)}/T_{ij}(p)$; otherwise $\overline{\mu_j(p)} = \mu_i(p)\overline{T_{ij}(p)}/T_{ij}(p)$ for all $p \in U_i \cap U_j$.

The family $\mu_{\tilde{U}}$ is called a (-1,1)-differential relative to $\tilde{U}$. If it satisfies

$$\sup \{ \| |\mu_i \circ z_i^{-1}| \|_\infty | i \in I\} < 1,$$

where $\| \ \|_\infty$ denotes the L^∞-norm, it is a <u>Beltrami differential relative to</u> $\tilde{U}$. The Beltrami differentials $\mu_{\tilde{U}}$ and $\mu_{\tilde{V}}$ relative to $\tilde{U}$ and $\tilde{V}$, respectively, are called equivalent if $\mu_{\tilde{U}} \cup \mu_{\tilde{V}}$ is a Beltrami differential relative to $\tilde{U} \cup \tilde{V}$. An equivalence class of Beltrami differentials relative to dianalytic atlases of X is called a <u>Beltrami differential on</u> X.

Let $f\colon X \to Y$ be a homeomorphism of Klein surfaces. We call f

K-<u>quasiconformal</u>, for a finite $K \geq 1$, if for each $p \in X$ and for each local variable z about p we can choose a local variable w about $f(p)$ such that the mapping $w \circ f \circ z^{-1}$ is K-quasiconformal near $z(p)$; f is quasiconformal if it is K-quasiconformal for some finite K. If f is quasiconformal, then the smallest number K for which f is K-quasiconformal is called the maximal dilatation of f.

By the theory of plane quasiconformal mappings it is clear that a 1-quasiconformal mapping is an isomorphism of Klein surfaces, and if $f_1: X_1 \to X_2$ and $f_2: X_2 \to X_3$ are K_1- and K_2-quasiconformal mappings, respectively, then $f_2 \circ f_1$ is K_1K_2-quasiconformal.

Let $f: X \to Y$ be quasiconformal, and let $\tilde{U} = \{(U_i, z_i) | i \in I\}$ be a dianalytic atlas of X. Assume that each set $f(U_i)$, $i \in I$, is contained in some dianalytic chart of Y. Then for each $i \in I$ we can choose a local variable w_i on $f(U_i)$ such that $w_i \circ f \circ z_i^{-1}$ is quasiconformal. After that choice the functions

$$\mu_i = \frac{\partial(w_i \circ f \circ z_i^{-1})/\partial\bar{z}}{\partial(w_i \circ f \circ z_i^{-1})/\partial z}$$

clearly form a Beltrami differential relative to $\tilde{U}$; its equivalence class μ is called the Beltrami differential of f, and f is called a μ-quasiconformal mapping of X.

Conversely, given a Beltrami differential μ on X, we can solve the differential equation $\partial f/\partial\bar{z} = \mu\partial f/\partial z$ on every chart (U,z). Let f_1 and f_2 be homeomorphic solutions corresponding to (U_1,z_1) and (U_2,z_2), respectively. If $U_1 \cap U_2 \neq \phi$, then a formla computation shows that the homeomorphism $f_2 \circ f_1^{-1}$ is either analytic or antianalytic depending on $z_2 \circ z_1^{-1}$. It follows that $\tilde{U}^\mu = \{(U,f)\}$ is a dianalytic atlas for the topological surface X. X together with the corresponding dianalytic structure is a Klein surface which we denote by X^μ.

A mapping $f: X \to Y$, μ-quasiconformal on X, is an isomorphism of X^μ onto Y, and the identity mapping $X \to X^\mu$ is a μ-quasiconformal mapping of X. Hence we have the following result.

T h e o r e m 1.2. *Let* μ *be a Beltrami differential on a Klein surface* X. *There exist* μ-*quasiconformal mappings of* X. *If* f_1 *and*

f_2 *are both* μ*-quasiconformal, then there exists an isomorphism* $g: f_1(X) \to f_2(X)$ *such that* $f_2 = g \circ f_1$.

1.9. Consider a disk D', $\overline{D'} \subset D = \{z \mid |z| < 1\}$ and a sense preserving homeomorphism f of D onto itself. Let $f': D \to D$ be a homeomorphism with the following properties:

(i) $f' = f$ in $D - D'$.

(ii) f' is locally quasiconformal in D'.

(iii) If f is quasiconformal in a neighborhood of a point of $\partial D'$, then f' is also quasiconformal in that neighborhood.

(iv) If f is quasiconformal in a neighborhood of $\partial D'$, then f' is quasiconformal in that neighborhood.

It is not difficult to see that for any f there exists an f' (which is not uniquely determined) with the above properties (see [14], Lemma 1.1).

It is clear that in the above we can replace the sets D and D' by the sets $D \cap H$ and $D' \cap H$ and still find an f' with the properties (i)-(iv). Recall that H denotes the upper half-plane.

Using this auxiliary mapping we shall show that any homeomorphism $g: X \to Y$ of compact Klein surfaces is homotopic to a quasiconformal mapping.

Consider a dianalytic atlas $\{(U_1,z_1),(U_2,z_2),\ldots,(U_n,z_n)\}$ of X with the following properties:

(i) Each $z_i(U_i)$ is either D or $D \cap \overline{H}$.

(ii) Each $g(U_i)$ is contained in a dianalytic chart (V_i,w_i) of Y such that $w_i(g(U_i))$ is either D or $D \cap \overline{H}$.

(iii) Each mapping $w_i \circ g \circ z_i^{-1}$ is sense-preserving.

Let D', $\overline{D'} \subset D$, be such a disk that the sets $z_i^{-1}(D')$ or $z_i^{-1}(D' \cap \overline{H})$ form a covering of X. Having done all this define the sequence $(g_0,g_1,\ldots,g_n)$ as follows. Set $g_0 = g$. If g_{i-1} is defined, then define g_i to be g_{i-1} outside of U_i; in U_i first set $f = w_i \circ g_{i-1} \circ z_i^{-1}$ and then define g_i as $w_i^{-1} \circ f' \circ z_i$, where f' is the auxiliary function defined by f and having the properties (i)-(iv).

By construction each g_i is homotopic to g_{i-1}, whence g_n is homotopic to $g_0 = g$. g_n is also locally quasiconformal in X. Since X is compact, g_n is quasiconformal. Hence we have proved the following result.

L e m m a 1.2. *The homotopy class of a homeomorphism of compact Klein surfaces contains quasiconformal mappings.*

2. Uniformization of Klein surfaces

Let $\phi\colon \tilde{X} \to X$ be the universal covering surface (in the topological sense) of a Klein surface X. We can endow $\tilde{X}$ with a dianalytic structure requiring $\phi\colon \tilde{X} \to X$ to be a morphism of Klein surfaces. Then the group G of covering transformations turns out to be a subgroup of the group of automorphisms of the Klein surface $\tilde{X}$. Knowing the classical results concerning Riemann surfaces it is not difficult to study G and $\tilde{X}$, since $\tilde{X}$ is, in fact, the universal covering surface of the orienting double X_o of X.

In our applications, however, another covering of X plays a fundamental role. This covering we obtain using the complex double X_c of X.

2.1. Let us start recalling some classical results about the uniformization of Riemann surfaces without boundary.

By the universal covering surface of a Riemann surface S we mean a pair $(\tilde{S},\phi)$ satisfying the following conditions, where I denotes the unit interval:

(i) $\tilde{S}$ is a simply connected Riemann surface, and $\phi\colon \tilde{S} \to S$ is a locally conformal mapping.

(ii) If $\gamma\colon I \to S$ is a path and if $\phi(\tilde{p}_o) = \gamma(0)$, then there exists a path $\tilde{\gamma}\colon I \to \tilde{S}$ with $\tilde{\gamma}(0) = \tilde{p}_o$ and $\phi \circ \tilde{\gamma} = \gamma$.

The path $\tilde{\gamma}$ in (ii) is called the lifting of γ from the point $\tilde{p}_o$.

A simply connected Riemann surface without boundary is conformally equivalent with the Riemann sphere, $\hat{\mathbb{C}}$, the finite complex plane, $\mathbb{C}$, or with the unit disk, D. If $\tilde{S}$ is $\hat{\mathbb{C}}$, then also S is $\hat{\mathbb{C}}$. If $\tilde{S}$ is conformally equivalent with $\mathbb{C}$, then S is a torus or $\mathbb{C}$ or $\mathbb{C}$ punctured once. In other cases $\tilde{S}$ is conformally equivalent with D. Especially if S is compact and $g(S) > 1$, $\tilde{S} = D$. Since we will be interested in this case only, we will from now on always assume, unless otherwise stated, that all Riemann surfaces under consideration have

D as the universal covering surface.

2.2. The group G of conformal automorphisms g of the universal covering surface $\tilde{S} = D$ of S satisfying $\phi \circ g = \phi$ is called the covering transformation group of D over S. It acts properly discontinuously on D producing S as the quotient $S = D/G$.

The group of Möbius transformations fixing D contains G as a subgroup, and S being compact G is generated by $2g(S)$ hyperbolic Möbius transformations $A_1, B_1, A_2, B_2, \ldots, A_{g(S)}, B_{g(S)}$ subject to the single relation

$$(2.1) \qquad A_1 B_1 A_1^{-1} B_1^{-1} \ldots A_{g(S)} B_{g(S)} A_{g(S)}^{-1} B_{g(S)}^{-1} = \mathrm{Id}.$$

Hyperbolic Möbius transformations fixing a disk depend on three real parameters. They are the attracting and the repelling fixed points and the multiplier. Hence a set of generators for G depends on $6g(S)$ real parameters. The relation (2.1) reduces the number to $6g(S) - 3$, and since S determines G only up to conjugation by a Möbius transformation, it follows that the analytic structure of S depends on $6g(S) - 6$ real parameters, as is well known.

2.3. Let (D', ϕ') and (D'', ϕ'') be the universal covering surfaces of $S' = D'/G'$ and $S'' = D''/G''$, respectively. A continuous mapping $f\colon S' \to S''$ can be lifted to a continuous mapping $\tilde{f}\colon D' \to D''$ satisfying

$$(2.2) \qquad \phi'' \circ \tilde{f} = f \circ \phi'.$$

Any continuous mapping $\tilde{f}\colon D' \to D''$ satisfying (2.2) is called a lifting of f. If $\tilde{f}_1$ and $\tilde{f}_2$ are both liftings of f, then there exists a $g_0'' \in G''$ such that

$$(2.3) \qquad \tilde{f}_2 = g_0'' \circ \tilde{f}_1.$$

Conversely, any mapping $\tilde{f}_2$ of the form (2.3) is a lifting of f.

A lifting $\tilde{f}$ of f defines a homomorphism $\hat{f}\colon G' \to G''$ by the formula $\tilde{f} \circ g' = \hat{f}(g') \circ \tilde{f}$, $g' \in G'$. The lifting $\tilde{f}_1 = g_0'' \circ \tilde{f}$ of f defines an $\hat{f}_1\colon G' \to G''$ which satisfies

$$\hat{f}_1(g') = g_0'' \circ \hat{f}(g') \circ g_0''^{-1}, \quad g' \in G',$$

i.e. there exists an inner automorphism A of G'' such that

$$\hat{f}_1 = A \circ \hat{f}. \tag{2.4}$$

Now we choose to call two homomorphisms $\hat{f}$ and $\hat{f}_1$ of G' into G'' equivalent if there exists an inner automorphism A of G'' for which (2.4) is satisfied. Then all homomorphisms arising from one continuous mapping $f: S' \to S''$ are equivalent.

A homomorphism $\hat{f}: G' \to G''$ defined in the above manner by a lifting of f is said to be induced by f. It is clear that one homomorphism $G' \to G''$ may be induced by several mappings $S' \to S''$.

If $f_i: D_i/G_i \to D_{i+1}/G_{i+1}$, $i = 1,2$, are continuous, and if the lifting $\tilde{f}_i: D_i \to D_{i+1}$ of f_i determines $\hat{f}_i: G_i \to G_{i+1}$, $i = 1,2$, then $\tilde{f}_2 \circ \tilde{f}_1: D_1 \to D_3$ is a lifting of $f_2 \circ f_1$ determining $\hat{f}_2 \circ \hat{f}_1: G_1 \to G_3$.

2.4. We can now give applications to Klein surfaces. Let (X_c, π, σ) be the complex double of a non-classical Klein surface X. Then X_c is a Riemann surface; assume that it has the unit disc D as a universal covering surface. Let G be the corresponding covering transformation group.

The antianalytic involution σ of X_c can be lifted to an antianalytic homeomorphism $\tilde{\sigma}$ of D onto itself. $\tilde{\sigma}$ defines an isomorphism $\hat{\sigma}: G \to G$.

Since $\tilde{\sigma}^2$ is a lifting of σ^2 = the identity mapping, we have

$$\tilde{\sigma}^2 \in G. \tag{2.5}$$

And since $\hat{\sigma}$ is an isomorphism,

$$\tilde{\sigma}\, G\, \tilde{\sigma}^{-1} = G. \tag{2.6}$$

The group G and the mapping $\tilde{\sigma}$ generate a group $R = (G, \tilde{\sigma})$, which we call a <u>reflection group</u>. The group R acts properly discontinuously in D, and produces X as the quotient $X = D/R$.

Any Fuschian group G which acts in D and admits an antianalytic automorphism $\tilde{\sigma}$ of D with (2.5) and (2.6) satisfied is called a <u>symmetric Fuchsian group</u>. $\tilde{\sigma}$ is a <u>symmetry on</u> G. By previous considerations every non-classical Klein surface gives rise to a symmetric Fuchsian group.

Conversely, if G is a symmetric Fuchsian group acting in D such that D/G is a Riemann surface, then, $\tilde{\sigma}$ being a symmetry on G, $D/(G,\tilde{\sigma})$ is a Klein surface whose complex double is D/G.

2.5. A reflection in a hyperbolic straight line in D is an antianalytic involution of D. Conversely, every antianalytic involution of D is a reflection in a hyperbolic line in D, as can be verified by a tedious computation (see [4], Theorem 1.9.4). It follows that if $\tilde{\sigma}$ is an involution, it leaves a hyperbolic line in D pointwise fixed. If now $\tilde{\sigma}$ has fixed points in D, then also $\tilde{\sigma}^2$ has; hence $\tilde{\sigma}$ is, by (2.5), an involution of D. So, if $\tilde{\sigma}$ has one fixed point in D, then it is a reflection, and as such its fixed point locus is a whole line in D.

2.6. Consider again the complex double (X_c,π,σ) of a non-classical Klein surface X. Let (D,ϕ) be the universal covering surface of the Riemann surface $X_c = D/G$. The involution $\sigma\colon X_c \to X_c$ has fixed points if and only if $\partial X \neq \phi$; assume that this is the case.

A lifting $\tilde{\sigma}\colon D \to D$ of σ need not have fixed points, but if $\tilde{p} \in X_c$ is left fixed by σ, then

$$\tilde{\sigma}(\phi^{-1}(\tilde{p})) = \phi^{-1}(\tilde{p})$$

by (2.2). It follows that if $\tilde{p} \in \phi^{-1}(\tilde{p})$, there exists a $g_0 \in G$ such that $\tilde{\sigma}(\tilde{p}) = g_0(\tilde{p})$; hence $g_0^{-1} \circ \tilde{\sigma}$ is a lifting of σ fixing $\tilde{p}$.

If $\partial X \neq \phi$, then we can choose, by the above observation, such a generator $\tilde{\sigma}$ of the corresponding reflection group $R = (G,\tilde{\sigma})$ which has fixed points in D. By Section 2.5 $\tilde{\sigma}$ is then a reflection in a line in D.

In [13] R.J. Sibner has studied finitely generated reflection groups and shown that if a symmetry $\tilde{\sigma}$ of G is a reflection in a line L, then G has a fundamental domain symmetric with respect to L. Using the method of uniformization by Beltrami equations he showed further that we can choose generators A_i, B_i, $i = 1,2,\ldots,p$, and C_j, $j = 1,2,\ldots,q$, of G which satisfy

$$\hat{\sigma}(A_i) = B_i, \quad \hat{\sigma}(B_i) = A_i, \quad \hat{\sigma}(C_j) = C_j^{-1}$$

for all $i = 1,2,\ldots,p$ and $j = 1,2,\ldots,q$ ([13], Theorem 7.1).

The situation is more complicated if $\partial X = \phi$. Then σ does not have fixed points; hence a lifting $\tilde{\sigma}$ of σ cannot have fixed points (in D) either. From the considerations in 2.5 it follows then that $\tilde{\sigma}^2 \neq \mathrm{Id}$ for all liftings $\tilde{\sigma}$ of σ. In this case one can choose generators A_i, B_i, $i = 1,2,\ldots,g$, and C of G such that

$$\tilde{\sigma}^2 = C, \quad \hat{\sigma}(A_i) = B_i, \quad \hat{\sigma}(B_i) = C \circ A_i \circ C^{-1}$$

for all $i = 1,2,\ldots,g$ ([13], Theorem 7.1).

These results in conjunction with the considerations in 2.2 lead us to conjecture that a dianalytic structure of a compact non-classical Klein surface X depends on $3g(X) - 3$ real parameters. Below we will give this statement in a more precise form and prove it.

3. Lifting of mappings

3.1. Let (X_c, π, σ) and (X'_c, π', σ') be the complex doubles of non-classical Klein surfaces X and X', respectively. Let $f: X \to X'$ be continuous. Generally we cannot lift it to a continuous mapping between the complex doubles. This is, however, possible if f is a homeomorphism. Let us assume that such is the case.

To construct the lifting we will have to consider first the orienting doubles (X_o, π_o, σ_o) and $(X'_o, \pi'_o, \sigma'_o)$ of X and X'. The projections $\pi_o: X_o \to X$ and $\pi'_o: X'_o \to X'$ being local homeomorphisms the orienting doubles have good lifting properties. For example we know that a path can be lifted to the orienting double from any point lying over its initial point and that the lifting is unique once the initial point is fixed. Using this path lifting we can lift the homeomorphism $f: X \to X'$ to a homeomorphism $\tilde{f}: X_o \to X'_o$ satisfying

$$\pi'_o \circ \tilde{f} = f \circ \pi_o,$$

i.e. $\tilde{f}$ is a lifting of f. The lifting is not unique: $\sigma'_o \circ \tilde{f} = \tilde{f} \circ \sigma_o$ is also a lifting of f. These are the only homeomorphic liftings of f. Note that the above construction can be carried out because f is a homeomorphism, and that it is not possible for only continuous mappings.

Let $\tilde{f}$ be a homeomorphic lifting of f to a mapping between the orienting doubles. Clearly $\tilde{f}(\partial X_o) = \partial X'_o$. Since $\tilde{f} \circ \sigma_o = \sigma'_o \circ \tilde{f}$, $\tilde{f}$ defines a homeomorphism $\tilde{f}\colon X_c \to X'_c$ of the corresponding complex doubles. (Recall that X_c was obtained by gluing the symmetric points of ∂X_o together.) It satisfies

$$\pi' \circ \tilde{f} = f \circ \pi,$$

i.e. it is a lifting of f to a mapping between the complex doubles.

If $\tilde{f}\colon X_c \to X'_c$ is a homeomorphic lifting of f, then $\sigma' \circ \tilde{f} = \tilde{f} \circ \sigma$ is also one, and these are the only homeomorphic liftings of f. Note that if we require the lifting to be only continuous, there may be more of them.

Let (D,ϕ) and (D,ϕ') be the universal covering surfaces of $X_c = D/G$ and $X'_c = D/G'$, respectively. A lifting $\tilde{f}\colon X_c \to X'_c$ of f can be lifted to a homeomorphism $\tilde{\tilde{f}}\colon D \to D$ satisfying $\tilde{f} \circ \phi = \phi' \circ \tilde{\tilde{f}}$. And if $\tilde{\tilde{g}}\colon D \to D$ is a lifting of the mapping $\sigma' \circ \tilde{f} = \tilde{f} \circ \sigma$, then there exists a lifting $\tilde{\tilde{\sigma}}'\colon D \to D$ of $\sigma'\colon X'_c \to X'_c$ such that $\tilde{\tilde{g}} = \tilde{\tilde{\sigma}}' \circ \tilde{\tilde{f}}$.

Let now $R = (G,\tilde{\tilde{\sigma}})$ and $R' = (G',\tilde{\tilde{\sigma}}')$ be the reflection groups corresponding to X and X', respectively. The above considerations in conjunction with Section 2.3 yield the following result, where $\theta = \pi \circ \phi$ and $\theta' = \pi' \circ \phi'$.

L e m m a 3.1. *A homeomorphism* $f\colon X \to X'$ *admits a lifting to a homeomorphism* $\tilde{\tilde{f}}\colon D \to D$ *satisfying* $\theta' \circ \tilde{\tilde{f}} = f \circ \theta$. *If* $\tilde{\tilde{g}}$ *is another such lifting, then there exists an* $r'_o \in R'$ *such that* $\tilde{\tilde{g}} = r'_o \circ \tilde{\tilde{f}}$. *Conversely, any mapping* $\tilde{\tilde{g}}$ *of that form is a lifting of* f.

3.2. A lifting $\tilde{\tilde{f}}\colon D \to D$ of f has the property that it takes R-equivalent points of D to R'-equivalent points. Conversely, any mapping $\tilde{\tilde{g}}\colon D \to D$ with this property induces a mapping $g\colon D/R \to D/R'$, and if $\tilde{\tilde{g}}$ is continuous, g is as well and $\tilde{\tilde{g}}$ is a lifting of g. This observation leads to the following useful result.

T h e o r e m 3.1. *If* $X = D/R$, *then the Klein surface* $X^* = (\hat{\mathbb{C}} - \bar{D})/R$ *is isomorphic with* X.

Proof. The mapping $z \to 1/\bar{z}$ maps D onto $\hat{\mathbb{C}} - \bar{D}$ and induces an isomorphism between X and X^*.

3.3. Consider a Beltrami differential μ on a Klein surface

$X = D/R$, where $R = (G,\tilde{\sigma})$. μ defines a Beltrami differential on D; we denote it by $\tilde{\mu}$. $\tilde{\mu}$ on D is a <u>Beltrami differential of the group</u> R, i.e. it satisfies

$$\tilde{\mu} = (\tilde{\mu} \circ g)\frac{\overline{\partial g/\partial z}}{\partial g/\partial z}$$

for all $g \in G$ and

$$\overline{\tilde{\mu}} = (\tilde{\mu} \circ \tilde{\sigma})\frac{\overline{\partial\tilde{\sigma}/\partial\bar{z}}}{\partial\tilde{\sigma}/\partial\bar{z}}.$$

If $\tilde{\mu}$ is a Beltrami differential of $R = (G,\tilde{\sigma})$ and $w: D \to w(D)$ a $\tilde{\mu}$-quasiconformal mapping, then $w \circ g \circ w^{-1}$ is conformal for each $g \in G$, and $w \circ \tilde{\sigma} \circ w^{-1}$ is anti-conformal. The elements $w \circ g \circ w^{-1}$, $g \in G$, form a quasi-Fuchsian group G^{μ} which together with $w \circ \tilde{\sigma} \circ w^{-1}$ generates a group R^{μ} called a <u>quasi-reflection group</u>. From the theory of quasi-Fuchsian groups it is clear that $w(D)/R^{\mu}$ is a Klein surface and that the mapping $w: D \to w(D)$ induces a μ-quasiconformal mapping $D/R \to w(D)/R^{\mu}$. These observations lead to the following version of simultaneous uniformization of Klein surfaces.

T h e o r e m 3.2. *Let* X *and* Y *be topologically equivalent compact Klein surfaces. Assume that* $g(X) > 1$. *Then there exists a quasi-reflection group* R *acting in some domain* B *such that* $X = B/R$ *and* $Y = (\hat{\mathbb{C}} - \overline{B})/R$.

Proof. Since $g(X) > 1$ we can find a reflection group R_o such that $X = D/R_o$. By Lemma 1.2 we can find a quasiconformal mapping $f: X \to Y$; let μ be the Beltrami differential of f. Lift μ to a Beltrami differential $\tilde{\mu}$ of R_o and continue it by 0 to the whole $\hat{\mathbb{C}}$. Then $\tilde{\mu}$ is a Beltrami differential of R_o also in $\hat{\mathbb{C}} - \overline{D}$. Let w be a $\tilde{\mu}$-quasiconformal mapping of the plane. Then $R = \{w \circ r \circ w^{-1} | r \in R_o\}$ is a quasireflection group; if $B = \hat{\mathbb{C}} - \overline{w(D)}$, then $X = B/R$ and $Y = (\hat{\mathbb{C}} - \overline{B})/R$. The theorem is proved.

4. Homotopic mappings

4.1. Consider non-classical Klein surfaces $X = D/R$, $R = (G,\tilde{\sigma})$, and $Y = D/R'$, $R' = (G',\tilde{\sigma}')$, and a homeomorphism $f\colon X \to Y$. Let $\tilde{f}\colon X_c \to Y_c$ be a lifting of f to a mapping between the complex doubles, and let further $\tilde{\tilde{f}}\colon D \to D$ be a lifting of $\tilde{f}$ (and of course of f). Following the classical considerations we see that $\tilde{\tilde{f}}$ induces an isomorphism $\hat{f}\colon G \to G'$ by the formula

$$\tilde{\tilde{f}} \circ g = \hat{f}(g) \circ \tilde{\tilde{f}}, \quad g \in G.$$

Consider the mapping $\tilde{\tilde{f}} \circ \tilde{\sigma}$. It is a lifting of the mapping $\tilde{f} \circ \sigma = \sigma' \circ \tilde{f}$. It follows that we can choose a generator $\tilde{\sigma}'$ of R' for which

$$\tilde{\tilde{f}} \circ \tilde{\sigma} = \tilde{\sigma}' \circ \tilde{\tilde{f}}.$$

Hence the isomorphism $\hat{f}\colon G \to G'$ can be extended to an isomorphism $\hat{f}\colon R \to R'$ such that for all $r \in R$

$$\tilde{\tilde{f}} \circ r = \hat{f}(r) \circ \tilde{\tilde{f}}, \quad \text{and} \quad \hat{f}(\tilde{\sigma}) = \tilde{\sigma}'.$$

The isomorphism $\hat{f}\colon R \to R'$ is not uniquely determined by f. Let $\tilde{g} = r_0' \circ \tilde{\tilde{f}}$ be another lifting of f. Then for an $r \in R$,

$$\tilde{g} \circ r = r_0' \circ \tilde{\tilde{f}} \circ r = r_0' \circ \hat{f}(r) \circ \tilde{\tilde{f}} = r_0' \circ \hat{f}(r) \circ r_0'^{-1} \circ \tilde{g}.$$

Hence $\tilde{g}$ defines an isomorphism $\hat{g}\colon R \to R'$ satisfying

$$\hat{g}(r) = r_0' \circ \hat{f}(r) \circ r_0'^{-1}, \quad r \in R,$$

i.e. there exists an inner automorphism A of R' such that

$$(4.1) \qquad \hat{g} = A \circ \hat{f}.$$

As in the case of Riemann surfaces we choose to call two isomorphisms $\hat{f}$ and $\hat{g}$ of R onto R' equivalent if $\hat{g} \circ \hat{f}^{-1}$ is an inner automorphism of R'. The relation is an equivalence relation. It is clear that all isomorphisms arising from one homeomorphism between the corresponding Klein surfaces are equivalent.

We also have the following technical result.

L e m m a 4.1. *If* $f: X \to X'$, $i = 0,1$, *are homeomorphisms inducing equivalent isomorphisms* $R \to R'$, *then there exist liftings* $\tilde{f}_i: D \to D$ *of the mappings* f_i, $i = 0,1$, *such that* $\hat{f}_0 = \hat{f}_1$.

It is left to the patient reader to prove this.

4.2. Since we shall be interested in the compact case, we will, from now on, always assume that Klein surfaces under consideration are compact.

Let X and X' be compact and $f_i: X \to X'$, $i = 0,1$, be homeomorphisms which induce equivalent isomorphisms $R \to R'$. Let $\tilde{f}_0$ and $\tilde{f}_1$ be those liftings of f_0 and f_1, respectively, which define the same isomorphism $\hat{f}_0 = \hat{f}_1$ of R onto R'.

The element of length of the hyperbolic metric in the unit disc, $|dz|/(1 - |z|^2)$, is invariant under the elements of a reflection group acting in D. Using this metric we define a homotopy between f_0 and f_1 in the following way. Let $z \in D$, and choose a number t, $0 \leq t \leq 1$. Define $\tilde{F}(z,t)$ as the point in D dividing the hyperbolic segment from $\tilde{f}_0(z)$ to $\tilde{f}_1(z)$ in the ratio $t: (1-t)$. Then $\tilde{F}: D \times I \to D$, $(z,t) \to \tilde{F}(z,t)$, is a homotopy between $\tilde{f}_0$ and $\tilde{f}_1$. Let $r \in R$. Since $\hat{f}_0(r) = \hat{f}_1(r)$, it follows from the invariance of the hyperbolic metric that $\tilde{F}(r(z),t) = \hat{f}_0(r)(\tilde{F}(z,t))$ for all $z \in D$ and for all t, $0 \leq t \leq 1$. This means that $\tilde{F}$ defines a continuous mapping $F: X \times I \to X'$ with the property $F \circ \tilde{\theta} = \theta' \circ \tilde{F}$, where $\tilde{\theta}$ denotes the projection $D \times I \to X \times I$ and θ' the projection $D \to X'$. Since $f_0(p) = F(p,0)$ and $f_1(p) = F(p,1)$ for all $p \in X$, f_0 and f_1 are homotopic. The homotopy F between f_0 and f_1 has the property that each mapping $F(\ ,t)$: $X \to X'$ maps ∂X into $\partial X'$.

Hence we have proved that homeomorphisms f_i, $i = 0,1$, which induce equivalent isomorphisms between the corresponding reflection groups, are homotopic. In order to prove the converse we need the following result.

L e m m a 4.2. *Let* X *be a compact Klein surface for which* $g(X) \geq 2$. *If* $f: X \to X$ *is a homeomorphism which is homotopic to the* *identity mapping of* X, *then* f *can be lifted to a mapping* $\tilde{f}: X_c \to X_c$ *which is homotopic to the identity mapping of* X_c.

Proof. For the following details I am indebted to P. Tukia. Let us consider the case where X is non-orientable. Let X_o be the orienting double of X. We can lift the mapping $f: X \to X$ to a mapping $\tilde{f}: X_o \to X_o$ which is homotopic to the identity mapping of X_o. If $\partial X_o = \emptyset$, $X_o = X_c$, and we have nothing to prove. Assume that $\partial X_o \neq \emptyset$.

We can find a discontinuous group G_o, acting in D, whose region of discontinuity is Ω, for which $X_o = (\Omega \cap \overline{D})/G_o$. Let $E = \Omega \cap \overline{D}$ and $B = \Omega \cap \partial D$. The group G_o is of the second kind, i.e. its limit set $L = \partial D - B$ is nowhere dense in ∂D. Consider a lifting $\overset{\approx}{f}: E \to E$ of $\tilde{f}: X_o \to X_o$. We want to show that $\overset{\approx}{f}: E \to E$ admits a homeomorphic extension to the whole $\overline{D}$. In order to do this consider the mapping $\varphi = \overset{\approx}{f}|B$. Let $p_0 \in B$ be fixed. The set $\partial D - \{p_0\}$ can be interpreted as an interval and as such we can give an order in there. This order induces an order in the set $B - \{p_0\}$. In the same way we can also order the set $B - \{\overset{\approx}{f}(p_0)\}$. Since $\overset{\approx}{f}: E \to E$ is a homeomorphism it follows that $\varphi = \overset{\approx}{f}|B$ either preserves or reverses the order. Since $L = \partial D - B$ does not contain intervals, it follows, in both cases, that φ admits a homeomorphic extension $\tilde{\varphi}$ to a mapping of ∂D. It is not difficult to check now that setting $\overset{\approx}{f}(p) = \tilde{\varphi}(p)$ for all $p \in L$ we get a homeomorphic extension of $\overset{\approx}{f}$ to a mapping of $\overline{D}$. In the sequel we shall denote this extended mapping also by $\overset{\approx}{f}$. This topological result is due to P. Tukia ([15], Lemma 3).

Since $\tilde{f}: X_o \to X_o$ is homotopic to the identity mapping of X_o, the above lifting can be chosen in such a way that $\overset{\approx}{f} \circ g = g \circ \overset{\approx}{f}$ for all $g \in G_o$ (see [5], Theorem 6A). From this formula it follows that the fixed points of the non-identity elements of G_o are left fixed by $\overset{\approx}{f}$. Hence $\overset{\approx}{f}$ keeps the limit set L of G_o point-wise fixed.

We have assumed that $g(X) \geq 2$. Hence the limit set of G_o contains more than two points. Let α be a component of B. Since $\overset{\approx}{f}$ agrees with the identity mapping on the limit set of G_o, it follows that $\overset{\approx}{f}$ maps α onto itself. Hence we can use Earle's construction ([8]) to find a homotopy $\tilde{F}: X_o \times I \to X_o$ between the identity mapping of X_o and the mapping $\tilde{f}: X_o \to X_o$, which satisfies the following conditions:

(i) $\tilde{F}(\partial X_o, t) \subset \partial X_o$ for all t, $0 \leq t \leq 1$.

(ii) $\tilde{F}(\sigma_o(p),t) = \sigma_o(\tilde{F}(p,t))$, $p \in X_o$, where σ_o is the antianalytic involution of X_o for which $X = X_o/\sigma_o$.

Recall now that X_c can be obtained from X_o by means of gluing the symmetric points of ∂X_o together. Then we can deduce from (i) and (ii) that $\tilde{F}\colon X_o \times I \to X_o$ induces a mapping $\tilde{F}\colon X_c \times I \to X_c$ which is a homotopy between the identity mapping of X_c and a lifting $\tilde{f}$ of f. This proves the lemma for a non-orientable X. The same argument can be applied also to an orientable Klein surface.

Let $f_i\colon X \to X'$, $i = 0,1$, be homotopic homeomorphisms of compact Klein surfaces. Assume that $g(X) \geq 2$. Let $\tilde{f}_i\colon X_c \to X'_c$, $i = 0,1$, be liftings of the mappings f_0 and f_1. Then $\tilde{f}_1^{-1} \circ \tilde{f}_0$ is a lifting of $f_1^{-1} \circ f_0$. By Lemma 4.2 we can choose the above liftings in a way which makes $\tilde{f}_1^{-1} \circ \tilde{f}_0$ homotopic to the identity mapping of X_c. Hence $\tilde{f}_0$ and $\tilde{f}_1$ can be chosen to be homotopic. Let $\tilde{F}$ be a homotopy between $\tilde{f}_0$ and $\tilde{f}_1$ and let $\overset{\approx}{f}_0\colon D \to D$ be a lifting of $\tilde{f}_0$. We can lift the homotopy $\tilde{F}$ to a homotopy between $\overset{\approx}{f}_0\colon D \to D$ and some lifting $\overset{\approx}{f}_1\colon D \to D$ of $\tilde{f}_1$. Using this homotopy and the discontinuity of the group R' for which $X' = D/R'$ we can show that $\overset{\approx}{f}_0$ and $\overset{\approx}{f}_1$ define the same isomorphism $\hat{f}_0 = \hat{f}_1$ of R onto R', where R is the reflection group corresponding to X.

Collecting all the above results into a theorem we get the following:

T h e o r e m 4.1. *Let* $X = D/R$ *and* $X' = D/R'$ *be compact Klein surfaces. Assume that* $g(X) \geq 2$. *Two homeomorphisms* $f_i\colon X \to X'$ $i = 0,1$, *are homotopic if and only if they induce equivalent isomorphisms of* R *onto* R'.

4.3. For a later application it is convenient to express Theorem 4.2 in a more explicit form. Consider homotopic homeomorphisms f_0 and f_1 between compact Klein surfaces $X = D/R$ and $X' = D/R'$, $R = (G,\tilde{\sigma})$, $R' = (G',\tilde{\sigma}')$. By Theorem 4.1 we can choose liftings $\overset{\approx}{f}_0\colon D \to D$ and $\overset{\approx}{f}_1\colon D \to D$ of f_0 and f_1 for which $\hat{f}_0 = \hat{f}_1$. Since X_c is compact the limit set of G is ∂D, and we can again use a result of Tukia ([15], Lemma 3) and continue the mappings $\overset{\approx}{f}_0$ and $\overset{\approx}{f}_1$ to homeomorphisms of $\overline{D}$ onto itself. Considering the fixed points of the elements of G we can show, as before, that $\overset{\approx}{f}_0$ and $\overset{\approx}{f}_1$ must agree on the limit set of G, i.e. they agree on ∂D.

On the other hand, if $\tilde{\tilde{f}}_0$ and $\tilde{\tilde{f}}_1$ agree on ∂D, they define the same isomorphism of R onto R'. Hence we have the following result.

T h e o r e m 4.2. *Let* $X = D/R$ *and* $X' = D/R'$ *be compact Klein surfaces. Assume that* $g(X) \geq 2$. *Two homeomorphisms* $f_i \colon X \to X'$, $i = 0,1$, *are homotopic if and only if there exist liftings* $\tilde{\tilde{f}}_i \colon \overline{D} \to \overline{D}$ *of the mappings* f_i, $i = 0,1$, *which agree on* ∂D.

4.4. Theorem 4.2 has an interesting topological corollary. Consider homeomorphisms $f_i \colon X \to X'$, $i = 0,1$, of compact Klein surfaces of genus at least 2. We have proved that if f_0 and f_1 are homotopic, there exist homotopic liftings $\tilde{f}_0 \colon X_c \to X_c'$ and $\tilde{f}_1 \colon X_c \to X_c'$ of f_0 and f_1. The converse is also true.

L e m m a 4.3. *Assume that* X *and* X' *are compact Klein surfaces and that* $g(X) \geq 2$. *Two homeomorphisms* $f_i \colon X \to X'$, $i = 0,1$, *are homotopic if and only if there exist homotopic liftings* $\tilde{f}_i \colon X_c \to X_c'$, $i = 0,1$.

Proof. In view of Section 4.2 it suffices to show the "if"-part of the lemma.

Assume that the liftings $\tilde{f}_0 \colon X_c \to X_c'$ and $\tilde{f}_1 \colon X_c \to X_c'$ are homotopic. Then, by Theorem 4.2, there exist liftings $\tilde{\tilde{f}}_0 \colon \overline{D} \to \overline{D}$ and $\tilde{\tilde{f}}_1 \colon \overline{D} \to \overline{D}$ of $\tilde{f}_0$ and $\tilde{f}_1$ which agree on ∂D. Since these mappings are also liftings of the mappings $f_i \colon X \to X'$, $i = 0,1$, we can again employ Theorem 4.2 to see that f_0 and f_1 are homotopic.

R e m a r k. The above results are not true if $g(X) = 0$ or $g(X) = 1$.

5. Spaces of Klein surfaces

5.1. Consider a fixed compact two-manifold Z, which may have boundary. As was realized by N.L. Alling and N. Greenleaf ([4], Theorem 1.7.1), Z can be endowed with a dianalytic structure. This structure is not unique. In this chapter we will give a parametrization for the dianalytic structures of Z. We shall see that the classical methods work on Klein surfaces. The cases $g(Z) = 0$ and $g(Z) = 1$ being well understood ([4], 1.9) we consider only the general case $g(Z) \geq 2$.

Let $K(Z)$ denote the set of dianalytic structures on Z. In $K(Z)$ we define the following relations:

(i) $X_1 \sim X_2$ if the homotopy class of the identity mapping $X_1 \to X_2$ contains an isomorphism of Klein surfaces.

(ii) $X_1 \approx X_2$ if there exists an isomorphism $X_1 \to X_2$ which is homotopic to the identity mapping of Z modulo ∂Z (fixing ∂Z point-wise).

It is clear that the relations $\sim$ and $\approx$ are equivalence relations.

D e f i n i t i o n 5.1. $T(Z) = K(Z)/\sim$ *is the reduced Teichmüller space of* Z, *and* $T_B(Z) = K(Z)/\approx$ *is the Bers' Teichmüller space of* Z.

We shall here be mainly interested in the reduced Teichmüller space of Z which will later be referred to as the Teichmüller space of Z. Our technique can be, however, applied also to $T_B(Z)$.

5.2. In $T(Z)$ we define the Teichmüller metric d in the usual way: If $X_i \in p_i \in T(Z)$, $i = 1,2$, then define

$$d(p_1,p_2) = \frac{1}{2} \inf \{\log K_f | f \in F\}, \tag{5.1}$$

where F denotes the class of quasiconformal mappings $X_1 \to X_2$ homotopic to the identity mapping of Z, and K_f denotes the maximal dilatation of f.

In $T_B(Z)$ we define the Teichmüller metric in the same way. The class F consists then of those quasiconformal mappings $X_1 \to X_2$ which are homotopic to the identity mapping modulo ∂Z.

The above definition does not depend on the choice of Klein surfaces representing the points p_1 and p_2. The fact that d is really a metric can be proved exactly in the classical way.

5.3. As in the case of Riemann surfaces $T(Z)$ with the metric (5.1) is a manifold homeomorphic to a domain in a Euclidean space. To prove this we will fix a point $p = [X] \in T(Z)$ and construct a local variable about it which turns out to be a homeomorphism of $T(Z)$ onto a domain in a Euclidean space.

For our constructions it is convenient first to express $T(Z)$ in terms of Beltrami differentials on X. Let X_1 and X_2 be Klein surfaces in $K(Z)$. Use Lemma 1.2 to find quasiconformal mappings $f_i \colon X \to X_i$, $i = 1,2$, homotopic to the identity mapping. Let μ_i denote the complex dilatation of f_i. Then, by Theorem 1.2, X_i and X^{μ_i} correspond to the same point in $T(Z)$, $i = 1,2$. Hence $X_1 \sim X_2$ if and only if $X^{\mu_1} \sim X^{\mu_2}$. In view of this observation $T(Z)$ can be defined in terms of Beltrami differentials of X as follows.

In the set $B(X)$ of Beltrami differentials of X define the relation $\sim$ setting $\mu_1 \sim \mu_2$ if the homotopy class of the identity mapping $X^{\mu_1} \to X^{\mu_2}$ contains an isomorphism of Klein surfaces. Then $\sim$ is clearly an equivalence relation in $B(X)$. The quotient

$$T(Z,X) = B(X)/\sim$$

is called the <u>Teichmüller space of Z relative to X</u>. The definition of the Teichmüller metric can be transformed into $T(Z,X)$ and we get:

L e m m a 5.1. *The natural mapping* $T(Z) \to T(Z,X)$ *is an isometric homeomorphism.*

Let us call the point in $T(Z,X)$ which is represented by 0, the origin of $T(Z,X)$. Consider a point $p = [\mu] \in T(Z,X)$. It is not difficult to verify that the mapping $\gamma\colon I \to T(Z,X)$, $t \to [t\mu]$, is a continuous mapping with $\gamma(0) = 0$ and $\gamma(1) = p$. Hence $T(Z,X)$ is connected. In the same way one can show that $T(Z,X)$ is simply connected. The details here are exactly as in the classical case.

5.4. To find a connection between the Teichmüller space and the space of quadratic differentials we can employ a classical method.

Since everything is well known in the classical case we will here consider only non-classical Klein surfaces. We have previously assumed that $g(X) \geq 2$. Hence we can write $X = D/R$, where $R = (G,\tilde{\sigma})$ is a reflection group. Recall that the Beltrami differentials $\mu \in B(X)$ of X can be interpreted as the Beltrami differentials $\tilde{\mu}$ of the group R in D.

For a Beltrami differential $\tilde{\mu}$ of R in D let f_μ denote the unique $\tilde{\mu}$-quasiconformal mapping of $\overline{D}$ onto itself which keeps $1,i,-1$ fixed. We call f_μ the normalized $\tilde{\mu}$-quasiconformal mapping of D. The following lemma is a corollary of Theorem 4.2.

L e m m a 5.2. $\mu_1 \sim \mu_2$ *if and only if* $f_{\mu_1}|\partial D = f_{\mu_2}|\partial D$.

Following Bers' construction we introduce now, for each Beltrami differential $\tilde{\mu}$ of R, the unique normalized quasiconformal mapping f^{μ} of $\hat{\mathbb{C}}$ which has the complex dilatation $\tilde{\mu}$ in D, is conformal in $D^* = \hat{\mathbb{C}} - \overline{D}$, and keeps $1, i, -1$ fixed. Imitating the classical reasoning we get the following result from Lemma 5.2.

L e m m a 5.3. *$\mu_1 \sim \mu_2$ if and only if $f^{\mu_1}|D^* = f^{\mu_2}|D^*$.*

5.5. For the reader's convenience we recall in this section some properties of Schwarzian derivatives.

Let f be any locally schlicht meromorphic function which is defined in a simply connected domain A. Then its Schwarzian derivative

$$S_f = f'''/f' - \frac{3}{2}(f''/f')^2$$

is holomorphic in A.

By a direct computation one can check that $S_f = 0$ if and only if f is a Möbius transformation. For the composite function we have

$$S_{f \circ g} = (S_f \circ g)g'^2 + S_g. \tag{5.2}$$

And for the function h, $h(z) = \overline{f(\overline{z})}$, we have

$$S_h(z) = \overline{S_f(\overline{z})}. \tag{5.3}$$

If φ is holomorphic in D^*, define

$$||\varphi|| = \sup\{|\varphi(z)|(|z|^2 - 1)^2 | z \in D^*\}, \tag{5.4}$$

and denote

$$H(D^*) = \{\varphi | \varphi \text{ holomorphic in } D^* \text{ and } ||\varphi|| < \infty\}.$$

Then $H(D^*)$ with the norm (5.4) is a Banach space.

5.6. Let us return to the Teichmüller space $T(Z,X)$ of Z relative to $X = D/R$. By Theorem 3.1, $X = D^*/R$. An element φ of $D_o^2(X)$ can be, in a natural way, interpreted as a holomorphic function φ in D^* satisfying

$$\varphi = (\varphi \circ g)g'^2 \tag{5.5}$$

for all $g \in G$ and

(5.6) $\overline{\varphi} = (\varphi \circ \tilde{\sigma})(\partial\tilde{\sigma}/\partial\bar{z})^2$.

From the compactness of X it follows that $\varphi \in H(D^*)$.

Conversely, any $\varphi \in H(D^*)$ satisfying (5.5) and (5.6) arises from a quadratic differential of X and is called a quadratic differential of the group R. Denote the set of holomorphic quadratic differentials of R by $D_o^2(R)$. Then $D_o^2(R)$ is a subspace of the Banach space $H(D^*)$. If X is a non-classical Klein surface, then $D_o^2(R)$ is a real Banach space; otherwise it is a complex Banach space.

Let $\tilde{\mu}$ be a Beltrami differential of R in D. Then, by (5.2) and (5.3), $S_{f^\mu|D^*} \in D_o^2(R)$. Denote

$$\Delta(R) = \{S_{f^\mu|D^*} \mid \tilde{\mu} \text{ a Beltrami differential of } R \text{ in } D\}.$$

Then $\Delta(R) \subset D_o^2(R)$, and in view of Lemma 5.3 we obtain a bijective mapping

$$\psi\colon T(Z,X) \to \Delta(R), \quad [\mu] \to S_{f^\mu|D^*}.$$

It is well known that in the classical case ψ is a homeomorphism onto its image which is an open set in the Banach space of quadratic differentials of X. We wish to prove that the same holds also in the non-classical case.

5.7. Assume that Z is non-classical, i.e. Z is non-orientable or has a non-empty boundary. For the topological surface Z we can, of course, construct a complex double which is a triple (Z_c,π,σ) satisfying the following conditions:

(i) Z_c is an orientable two-manifold without boundary.

(ii) The projection $\pi\colon Z_c \to Z$ is an unramified double covering map.

(iii) $\sigma\colon Z_c \to Z_c$ is a sense reversing involution satisfying $\pi = \pi \circ \sigma$.

Let (Z_c,π,σ) be fixed. Then any dianalytic structure $X \in K(Z)$ induces an analytic structure X_c on Z_c such that (X_c,π,σ) is the complex double of the Klein surface X. In the sequel we shall assume that all complex doubles of Klein surfaces $X \in K(Z)$ are obtained in the above manner.

If X and X' represent the same point in $T(Z)$, then there exists an isomorphism $f\colon X \to X'$ which is homotopic to the identity mapping of Z. By Lemma 4.3 f can be lifted to an isomorphism $\tilde{f}\colon X_c \to X'_c$ homotopic to the identity mapping of Z_c. Hence X_c and X'_c correspond to the same point in $T(Z_c)$. It follows that $\pi^*([X]) = [X_c]$ defines a mapping $\pi^*\colon T(Z) \to T(Z_c)$.

To prove that π^* is one-to-one choose two Klein surfaces X and X' from $K(Z)$, and assume that $\pi^*([X]) = \pi^*([X'])$. This means that $X_c \sim X'_c$. Hence there exists an isomorphism $\tilde{f}\colon X_c \to X'_c$ which is homotopic to the identity mapping of Z_c. Since the involutions σ and σ' of the complex doubles X_c and X'_c, respectively, agree as mappings of Z_c, it follows that the isomorphism $\sigma'^{-1} \circ \tilde{f} \circ \sigma$ is also homotopic to the identity mapping. Since the homotopy class of a homeomorphism of compact Klein surfaces contains, by Theorem 4.2, at most one isomorphism, it follows that $\sigma'^{-1} \circ \tilde{f} \circ \sigma = \tilde{f}$, i.e.

$$\sigma' \circ \tilde{f} = \tilde{f} \circ \sigma.$$

Hence $\tilde{f}$ is a lifting of some isomorphism $f\colon X \to X'$ which is, by Lemma 4.3, homotopic to the identity mapping of Z. This means that $\pi^*\colon T(Z) \to T(Z_c)$ is one-to-one.

Let us prove now that $\pi^*\colon T(Z) \to T(Z_c)$ is isometric. To this end choose two points $[X]$ and $[X']$ from $T(Z)$. Let (X_c,π,σ) and (X'_c,π',σ') again be the corresponding complex doubles. Denote by $d_c([X_c],[X'_c])$ the distance of the point $[X_c]$ from the point $[X'_c]$ in $T(Z_c)$. Then clearly

$$d([X],[X']) \geq d_c([X_c],[X'_c]).$$

By the Teichmüller extremal mapping theorem we can find a unique quasiconformal mapping $\tilde{f}\colon X_c \to X'_c$ with the smallest maximal dilatation $K_{\tilde{f}}$ in the homotopy class of the identity mapping. Then $d_c([X_c],[X'_c]) = \frac{1}{2} \log K_{\tilde{f}}$.

Consider the mapping $\tilde{f}' = \sigma'^{-1} \circ \tilde{f} \circ \sigma\colon X_c \to X'_c$. Since σ and σ' coincide as mappings of Z_c, $\tilde{f}'$ is homotopic to the identity mapping. Since $K_{\tilde{f}'} = K_{\tilde{f}}$, it follows from the uniqueness of the Teichmüller extremal mapping that $\tilde{f}' = \tilde{f}$, i.e.

$$\tilde{f} \circ \sigma = \sigma' \circ \tilde{f}.$$

Hence $\tilde{f}$ is a lifting of some quasiconformal mapping $\tilde{f}\colon X \to X'$ which is, by Lemma 4.3, homotopic to the identity mapping. This shows that

$$d([X],[X']) \le d_c([X_c],[X_c']).$$

Hence we have shown that $d([X],[X']) = d_c([X_c],[X_c'])$, i.e. π^* is isometric.

Consider again the topological complex double (Z_c,π,σ) of Z. The automorphism σ of Z_c induces a mapping $\sigma^*\colon K(Z_c) \to K(Z_c)$ in the following way. If $\{(U_i,z_i) \mid i \in I\}$ is a dianalytic atlas of Z_c corresponding to the dianalytic structure Y of Z_c, then $\sigma^*(Y)$ is the dianalytic structure of Z_c defined by the atlas $\{(\sigma^{-1}(U_i), z_i \circ \sigma) \mid i \in I\}$.

It is easy to verify that the above mapping $\sigma^*\colon K(Z_c) \to K(Z_c)$ induces an involution $\sigma^*\colon T(Z_c) \to T(Z_c)$. And we have the following result.

T h e o r e m 5.1. *$T(Z)$ is homeomorphic to $\pi^*(T(Z))$, and* $\pi^*(T(Z)) = \{p \in T(Z_c) \mid \sigma^*(p) = p\}$.

Proof. By previous considerations π^* is a homeomorphism onto its image. Hence the first part of the theorem is already proved.

Let us denote the fixed-point set of the mapping σ^* of $T(Z_c)$ by $T^{\sigma}(Z_c)$ and prove that $\pi^*(T(Z)) = T^{\sigma}(Z_c)$. Note first that the inclusion $\pi^*(T(Z)) \subset T^{\sigma}(Z_c)$ follows immediately from the definitions. To prove the converse inclusion it is convenient to consider the spaces $T(Z,X)$ and $T(Z_c,X_c)$ instead of the spaces $T(Z)$ and $T(Z_c)$. Here X_c is the complex double of the Klein surface X.

Assume that $\sigma^*([Y]) = [Y] \in T(Z_c)$. Let $\alpha\colon X_c \to Y$ be the unique quasiconformal mapping which is homotopic to the identity mapping of Z_c and has the smallest maximal dilatation in its homotopy class. Let μ be the complex dilatation of α. Then the lifting $\tilde{\mu}$ of μ is a Beltrami differential of the covering group G corresponding to X_c. Let $R = (G,\tilde{\sigma})$ be the reflection group for which $X = D/R$. Our theorem is proved if we succeed in showing that $\tilde{\mu}$ is a Beltrami differential of R in D.

Since $\sigma^*([Y]) = [Y]$, there exists an isomorphism $g\colon Y \to \sigma^*(Y)$ which is homotopic to the identity mapping of Z_c. Note that the mapping $\sigma\colon \sigma^*(Y) \to Y$ is, by definition, an isomorphism. We obtain the following diagram

$$\begin{array}{ccccc} X_c & \xrightarrow{\alpha} & Y & & \\ \Big\downarrow \sigma & & \Big\downarrow f & \searrow g & \\ & & & & \sigma^*(Y), \\ X_c & \xrightarrow{\alpha} & Y & \swarrow \sigma & \end{array}$$

where $f = \sigma \circ g$ is an isomorphism. Since the mapping g is homotopic to the identity mapping of Z_c, $f \circ \alpha$ is homotopic to $\alpha \circ \sigma$. On the other hand, $f \circ \alpha$ and $\alpha \circ \sigma$ are both extremal mappings in their homotopy class. (This follows from the facts that α is the Teichmüller extremal mapping and f and σ are isomorphisms of Klein surfaces.) Hence they must agree, i.e. the above diagram is commutative. It follows that we can lift the mappings α, σ and f to mappings of D for which the equation

$$\tilde{\alpha} \circ \tilde{\sigma} = \tilde{f} \circ \tilde{\alpha}$$

is satisfied. From this equation we deduce that the complex dilatation $\tilde{\mu}$ of $\tilde{\alpha}$ satisfies

$$\overline{\tilde{\mu}} = (\tilde{\mu} \circ \tilde{\sigma}) \frac{\overline{\partial\tilde{\sigma}/\partial\bar{z}}}{\partial\tilde{\sigma}/\partial\bar{z}}.$$

Hence $\tilde{\mu}$ is a Beltrami differential of $R = (G, \tilde{\sigma})$ in D. This proves the theorem.

5.8. Let $X = D/R$, $R = (G, \tilde{\sigma})$, and $X_c = D/G$ as before. Repeating the construction of 5.6 we obtain a mapping

$$\psi_c \colon T(Z_c, X_c) \to \Delta(G).$$

Here $\Delta(G)$ is defined as $\Delta(R)$ in 5.6. The classical theory tells us that $\Delta(G)$ is open in $D_o^2(G)$ and that ψ_c is a homeomorphism.

Since $G \subset R$, $D_o^2(R) \subset D_o^2(G)$, and we obtain a commutative diagram

$$(5.7) \qquad \begin{array}{ccc} T(Z_c, X_c) & \xrightarrow{\psi_c} & \Delta(G) \\ \pi^* \Big\uparrow & & \Big\uparrow \text{incl.} \\ T(Z, X) & \xrightarrow{\psi} & \Delta(R) \end{array}$$

As was noted in Section 1.5, the involution σ of X_c defines an involution σ^* of $D_o^2(X_c) = D_o^2(G)$. A computation shows that the diagram

$$
\text{(5.8)} \qquad
\begin{array}{ccc}
T(Z_c,X_c) & \xrightarrow{\ \psi_c\ } & D_o^2(G) \\
\sigma^* \Big\uparrow & & \Big\uparrow \sigma^* \\
T(Z_c,X_c) & \xrightarrow{\ \psi_c\ } & D_o^2(G)
\end{array}
$$

is commutative. Considering the fixed point sets of the mappings σ^* we obtain then, by Theorem 5.1, the following formula:

$$
\text{(5.9)} \qquad \Delta(R) = \Delta(G) \cap D_o^2(R).
$$

Since, by the classical result, $\Delta(G)$ is open in $D_o^2(G)$, and since $D_o^2(R) \subset D_o^2(G)$, $\Delta(R)$ is, by (5.9) open in $D_o^2(R)$. The fact that ψ is a homeomorphism follows from the commutative diagram (5.7) and from the facts that $\pi^*\colon T(Z) \to \pi^*(T(Z))$ and ψ_c are homeomorphisms.

5.9. In $D_o^2(R)$ we have defined the norm (5.4). On the other hand, by Theorem 1.1, $D_o^2(X) = D_o^2(R)$ is a $(3g(X)-3)$-dimensional real vector space. Thus, in $D_o^2(R)$ we have also a Euclidean metric, which defines the same topology as the norm (5.4). Hence we get from the preceding considerations in conjunction with the classical results:

T h e o r e m 5.2. *Let* Z *be a compact topological two-manifold with* $g(Z) \geq 2$. *The reduced Teichmüller space* $T(Z)$ *of* Z *is a manifold homeomorphic to an open simply connected subset of a Euclidean space. If* Z *is orientable and does not have boundary, then* $\dim_{\mathbb{C}} T(Z) = 3g(Z)-3$, *otherwise* $\dim_{\mathbb{R}} T(Z) = 3g(Z)-3$.

5.10. Assume that Z is not classical. It is well-known that $T(Z_c)$ can be endowed with a natural complex structure such that ψ_c is a biholomorphic embedding of $T(Z_c)$ into the complex Banach space $D_o^2(G)$ ([7], Theorem I).

It is easy to check that the involution $\sigma^*\colon D_o^2(G) \to D_o^2(G)$ in diagram (5.8) is a conjugate-linear mapping. Hence it follows from (5.8) that $\sigma^*\colon T(Z_c,X_c) \to T(Z_c,X_c)$ is an anti-analytic involution. Thus $\pi^*(T(Z))$, being the fixed point locus of σ^*, is a real analytic submanifold of $T(Z_c)$. (The real analytic structure of $\pi^*(T(Z))$ can be defined by the real analytic atlas $\{(\pi^*(T(Z)),\psi_c)\}$.) Then we can transform the real analytic structure of $\pi^*(T(Z))$ into a real analytic structure of $T(Z)$ using the mapping π^*. The atlas $\{(T(Z),\psi)\}$ is a real analytic atlas of the above structure. Finally, it is clear that the mapping $\pi^*: T(Z) \to T(Z_c)$ is a real analytic immersion.

Remark 1. Let H be the group generated by the involution $\sigma: Z_c \to Z_c$. Then the space $\pi^*(T(Z))$ is the relative Teichmüller space $T(Z_c,H)$ of Kuribayashi studied by Earle in [9].

Remark 2. Our methods could also be applied to the Bers' Teichmüller space $T_B(Z)$ of a non-orientable Z. The result seems to be that $T_B(Z)$ can be embedded into $T_B(Z_o)$ as a real analytic submanifold. Here Z_o is the orientable double of the topological surface Z.

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