PAST MASTERS AND THEIR MODERN FOLLOWERS

Hannu Nurmi

INTRODUCTION

It is in the very nature of a society that men must sometimes surrender some of their freedom of action to various kinds of collectives which then act in the name – in the case of democratic systems with the explicit authority – of the individuals involved. The social theory based on individualism has ever since Hobbes tried to give explanations for the emergence of these kinds of collectivities in terms of the perceived need to provide public goods for their members. Regardless of the reason for the existence of the collectivities acting as corporate actors, the members can pursue their individual interests in the collective bodies in two ways: (i) by trying to establish a decision making procedure which would guarantee that their interests are taken into account in the decisions made in the name of the collectivity, or (ii) by trying to influence the decisions by delegating the entire decision making power to suitable person or group of persons. If we approach the issue from the viewpoint of a benevolent institution-designer, the strategy (i) boils down to the problem of designing an optimal decision making procedure for direct democracy. The strategy (ii), on the other hand, pertains to the basic problem of representative systems:[1] what kind of persons would make the best decisions for the collectivity?

One must not, of course, forget that the vast majority of human beings have never had any chance whatsoever of influencing either the decision making procedures of the collectivities which they are members of, or the kinds of persons ruling them. Nevertheless, they all have had an interest in both of these things. Maybe the distinction between (i) and (ii) can be made clearer in the light of two basic views of representation to which they can be related. The view of representation which is closely related to (i) holds that the representative body should be a sort of minature of the population it represents, i.e. the crucial individual properties should be nearly identically distributed in the body and in the population. The view that seem to be related to (ii), in turn, is elitistic: the body should consist of the most competent members of the population.

Let us now turn to the works of two pioneers of the group decision and social choice theory keeping in mind the above distinction. As will hopefully become evident in the following, the distinction made above has played a role already at the very beginning of this today rapidly expanding field of inquiry.

Let us now go back in history to the days immediately preceding the great revolution of 1789. In those times Chevalier Jean-Charles de Borda (1733–1799) and Marie Jean Antoine Nicolas Caritat, Marquis de Condorcet (1743–1794) presented their main works in the group decision theory. In the present context it is neither possible nor important to give a detailed account of their life histories. We shall focus on their research strategies, main results and the problems left open by their inquiries. Thereafter, we shall give an over-view of the results of modern group decision to the problems raised by Borda and Condorcet. Finally, we shall touch upon the rational consensus method of Lehrer and Wagner which seems to synthetize the strategies (i) and (ii) when viewed from the angle of a benevolent institution-designer.

BORDA

On June 16th 1770 Jean-Charles de Borda gave a lecture at the meeting of the French Academy. The lecture was entitled »Sur la Forme des Elections». The publication history of it is curious and revealing of the working habits of the Academy in those days. It is also a clear demonstration that the present day complaints about the long delay in the publication of manuscripts are grossly out of place (see Black 1958, 178–180; Todhunter 1949, 432). The talk was delivered again in 1784 and was printed in the proceedings of the Academy carrying the title »Mémoire sur les Elections au Scrutin». It seems likely that the year of printing was 1784 even though the text was dated 1781. The reason why Borda's work was forgotten for fourteen years, is anybody's guess. It may be that the commentators assigned by the Academy to review the paper simply neglected their duty. All the same, the acceptance of the work in 1784 was as warm as a scientist proposing a practical reform could possibly hope for: the Academy soon adopted Borda's voting procedure in its elections of officers. The success of the procedure was not, however, of permanent nature. Around year 1800 there were reports according to which Napoleon Bonaparte was dissatisfied with the procedure and insisted on its abandonment which took place soon thereafter.

Both Borda and Condorcet were respected scientists in their own time despite the delay in the publication of Borda's work. Borda was a member of the Academy and an officer both in the navy and in the cavalry. His main accomplishments were in the design of measurement devices. He was also a member in the committee of the National Assembly that was to introduce the metric system. It may well be that the differences between the scholarly works of Borda and Condorcet are at least partly accountable by the fact that Borda was a man of practice with a background in the natural sciences, while Condorcet's disciplinary affiliations were mainly in the humanities.

In contradistinction to Condorcet, Borda never aimed at constructing a general theory of politics. The main motivation of his work in group decision theory was an undesirable property of the plurality voting procedure, nowadays called the Borda paradox (Granger 1956, 118–120). With the aid of the following example Borda was able to show that the plurality winner could well be an alternative that intuitively is not the best or most preferred one. Consider a voting body of 21 voters choosing from the set of three alternatives A, B and C using the plurality principle. Let us assume that 13 voters prefer B to A and the remaining 8 voters prefer A to B. Moreover, we assume that the above mentioned 13 voters prefer C to A, while the 8 voters have the opposite preference between A and C. Clearly, A would be defeated by both B and C by 13 votes against 8. Nevertheless, A could be the plurality winner. This is the case e.g. if we assume that in the group of 13 voters 7 prefer B to C and 6 prefer C to B. For example, the following preference configuration would correspond that case:[2]

7 voters	7 voters	6 voters	1 voter
A	B	C	A
C	C	B	B
B	A	A	C

The one man–one vote -principle gives the following result:

A	8 votes
B	7 votes
C	6 votes

Thus A wins by the plurality principle.

To get rid of this drawback of the plurality method Borda designed two voting methods which in closer scrutiny turn out to be logically related so that the latter of the methods is a special case of the former. This is duly noticed by Borda. The more general of the methods nowadays carries the name of its designer and is called the Borda count. It is a positional voting method defined for all n-tuples of individual strict (i.e. irreflexive, connected and transitive) preference orders. Every position in the preference orders is assumed to have the same amount of »merit» so that the alternative regarded as the worst by a voter receives the amount of a of merit from the voter; the next to worst alternative receives a + b, the next one a + 2b etc. The sum of merits given to an alternative by the voters is the Borda score of the alternative. The alternative (or alternatives) with the maximum Borda score is (are) the Borda winner (Borda winners).

In the above example assuming that a = b = 1 we get the following Borda scores:

A	37
B	42
C	47

Thus C is the Borda winner.

The other method proposed by Borda is based on comparing the total number of votes given to the alternatives in all pairwise comparisons. As was pointed out above, the result of the voting is necessarily the same as in the method outlined above (see also Black 1958, 158–159). In the above example A would get 8 votes when confronted with B and 8 votes when confronted with C, i.e. the total of 16 votes. B would get 13 + 8 = 21 votes. C, in turn, would get 13 + 13 = 26 votes. The result is identical with the one obtained by using the method outlined above with a=0 and b=1.

Borda's motivation was apparently to correct a specific drawback of the plurality method, *viz.* the fact the method does not guarantee the victory of an uncontestable alternative, *viz.* one that would defeat both of its contestants in pairwise comparisons. This criterion of winning has later on been called the Condorcet criterion and an alternative satisfying the criterion the Condorcet winner. An alternative is the Condorcet winner if it defeats all of its contestants in pairwise comparisons. Even though Borda never gave a name to this criterion of winning, his criticism of the plurality method boils down to the observation that this method cannot guarantee the choice of the Condorcet winner when one exists. Against this background it is strange that in his memoir Borda came up with a proposal for a voting procedure that also fails to guarantee the choice of the Condorcet winner. The first one to call attention to this weakness of the Borda count was Condorcet.

CONDORCET

Borda was a practical man who aimed at guaranteeing the proper working of a given institution – voting – starting from commonly accepted premises concerning man and society. He never critically evaluated the validity of these assumptions. Condorcet, on the other hand, was primarily a social philosopher. In his thinking the problems of social choice can be seen against the background of a global view of society. During all but the very last years of his life Condorcet was a widely respected and admired member of the upper nobility. He was one of the last encyclopedists. His political affiliation was with the Girondists. His general world view fit well together with that of the revolution and the rising bourgeois class. As it was for many other Girondists, the years immediately following the revolution were fatal to Condorcet. In the summer of 1793 his arrest was ordered. He managed to hide, however, for about a year until he was found and arrested in 1794. Shortly after that he died in prison under somewhat mysterious circumstances (Baker 1967).

Condorcet was affiliated with the Academy from 1769, being its permanent secretary from 1777 and a member from 1782. Therefore, he was aware of Borda's work. However, he was not present at the meeting in June 1770 when Borda gave his lecture on voting by the order of merit (Black 1958, 178 fn.). From the view-point of social choice theory Condorcet's main contribution is »Essai sur l'Application de l'Analyse à la Probabilité des Décisions Rendues à la Pluralité des Voix». In that work Condorcet points out that an anonymous geometrician (Borda) first observed the weakness of the plurality method and proposed an alternative new voting method. In the same context Condorcet adds that also the method proposed has the same weakness although Borda's method very rarely leads into trouble, i.e. does not choose the Condorcet winner (Condorcet 1785, clxxix; Black 1958, 179). Let us take a look at the example discussed by Condorcet:

30 voters	1 voter	10 voters	29 voters	10 voters	1 voter
A	A	C	B	B	C
B	C	A	A	C	B
C	B	B	C	A	A

Clearly A is the Condorcet winner. Let now the amount of merit of a first rank be d, of second rank e and of third rank f points. The Borda count is obviously a special case of this method. In the Borda count we have $f=a$, $e=a+b$ and $d=a+2b$. The following condition is then necessary for the choice of A (i.e. the Condorcet winner):

$31d + 39e + 11f > 39d + 31e + 11f$ or $8e > 8d$ which of course amounts to $e > d$. This condition guarantees that the choice is not B instead of A. But this condition means that the merit of the second rank is larger than that of the first rank, which is absurd. Obviously the points do not represent the preferences of the voters. Thus in this example no matter which specific values we give to d, e and f as long as the value assignment is consistent, the Condorcet winner will not be chosen.

Condorcet was not content with the mere criticism of the positional method of Borda, but outlined his own theory of voting in the essay mentioned above. From the point of view of modern social choice theory, the disciplinary affiliation that Condorcet gave to his theory is a strange one; he viewed voting procedures as part and parcel of probability theory. One should observe, however, that this was the prevailing conception of the nature of the theory. For example, both Borda and Marquis de Laplace shared this view. More specific to Condorcet was his idea of voting procedures as special cases of the so-called jury problem. In that problem one tries to determine the probability of a jury coming up with the right verdict when each juror has a fixed probability of being right. Condorcet approaches the collective decision making from the same angle and assumes that each voter has a fixed probability of being right. Given the electoral data on pairwise comparisons of alternatives, we can determine the probabilities of certain statements. These statements, in turn, express the preference relations between alternatives. By eliminating from the set of statements those with smaller probabilities, we can end up with a social preference order which is »right» in the deepest possible sense. We notice that Condorcet resorts to the strategy (ii) outlined in the introduction. Borda, on the other hand, clearly represents strategy (i). Anyway, Condorcet's way of looking at the problem is somewhat peculiar to a modern reader. Let us, therefore, illustrate it (see Black 1959, 164–165; Todhunter 1949, 353–375).

Let us assume that we have a completely homogeneous body of n voters faced with a dichotomous choice: vote for »yes» or »no». What is the probability that exactly m persons make the right decision when each voter has the probabality p of being right and the probability $1-p$ of being wrong? Obviously, the probability can be obtained from the binominal formula and is the following:

$$\frac{n!}{m!(n-m)!} p^m (1-p)^{n-m}.$$

Similarly the probability of exactly m persons' being wrong is

$$\frac{n!}{m!\,(n-m)!}\,p^{n-m}\,(1-p)^{m}.$$

If we do not know if the »yes» vote or »no» vote of exactly m persons is the right decision and $m > n-m$, we can calculate its probability of being the right one, as the ratio of the above probabilities is

$$\frac{p^{m}(1-p)^{n-m}}{p^{n-m}(1-p)^{m}}$$

which thus is the ratio between the probability of the right decision to the probability of the wrong decision when the group of voters voting similarly consists of m persons. Obviously

$$\frac{p^{m}(1-p)^{n-m}}{p^{m}(1-p)^{n-m}+p^{n-m}(1-p)^{m}}$$

is the probability that the decision made by the m persons is right. This can be expressed as follows:

$$(1)\quad \frac{p^{m}}{p^{m}+p^{n-m}(1-p)^{2mn-n}} = \frac{p^{2m-n}}{p^{2m-n}+(1-p)^{2m-n}}.$$

Similarly the probability of exactly m persons' being wrong is

$$(2)\quad \frac{(1-p)^{2m-n}}{p^{2m-n}+(1-p)^{2m-n}}.$$

Let us now concentrate on one of Condorcet's examples. The alternative set consists again of three alternatives A, B and C. The number of voters is 33. The data from pairwise comparisons can be presented in the following way (Black 1958, 169):

	A	B	C
A	–	18,15	18,15
B	15,18	–	32,1
C	15,18	1,32	–

where the left (right, respectively) element of the pair of numbers in i'th row and j'th column indicates the number of votes given to the alternative i (j) in the pairwise contest between i and j (i, j = A, B, C).

We notice immediately that A gets the majority of votes in each pairwise contest (and is, therefore, the Condorcet winner). In Condorcet's calculus A is not, however, chosen without further ado. Instead one compares the probabilities of the statements expressing preferences between alternatives. The probability that those 18 persons who voted for A in the pairwise contest between A and B are right is obtained by substituting $m=18$ and $n=33$ in the expression (1) above. Then we get the probability

$$\frac{p^3}{p^3+(1-p)^3}.$$

This is the probability of the statement $A > B$, i.e. the statement »A is better than B». Similarly the probability of the statement $A > C$ is

$$\frac{p^3}{p^3+(1-p)^3}.$$

When the »events» $A > B$ and $A > C$ are assumed to be mutually independent, we get

$$P(A > B \text{ and } A > C) = \frac{p^6}{p^6+2p^3(1-p)^3+(1-p)^6},$$

where P is, of course, the probability operator.

Let us compare this probability with the probability of another consistent preference, viz. $B > A$ and $B > C$. As we have assumed above that $m > n-m$, the probability $P(B > A)$ is the probability that the majority is wrong. Thus,

$$P(B > A) = \frac{(1-p)^3}{p^3+(1-p)^3}.$$

The probability of the statement $B > C$ is

$$P(B > C) = \frac{p^{31}}{p^{31}+(1-p)^{31}}.$$

Hence,

$$P(B > A \text{ and } B > C) = \frac{p^{31}(1-p)^3}{p^{34}+p^3(1-p)^{31}+p^{31}(1-p)^3+(1-p)^{34}}.$$

In Condorcet's calculus we cannot be sure even in this »obvious» case that the »right» alternative or candidate A has been chosen because in a certain interval [½, x] of values of p where $x<1$, the probability of the statement »B > A and B > C» is larger than the probability of the statement »A > B and A > C».

The example is even in Condorcet's view so obvious that despite the result yielded by his calculus he is prepared to propose the choice of A based on »straight-forward reasoning». Thus the case in which the stringent requirements of the winning criterion bearing Condorcet's name are fulfilled, is actually an exception to the general rule proposed by Condorcet. In this context Black observes that Condorcet's theory of voting procedures is largely independent of the probability calculus that has later on been criticized by several authors. The probability concept is still present in Condorcet's other investigations on situations in which there are cyclical majorities. As an example let us take the case in which the majority thinks that A > B, B > C and C > A. To determine the probabilities of the corresponding statements one would need to know the probabilities of the voters' being right. Condorcet points out, however, that it is possible to infer the probability of an alternative's or candidate's being the right choice even though one does not know the exact probabilities. The inference is based on an additional assumption, viz. that the larger the majority of voters voting according to a given preference statement, the larger is the probability of the statement. Thus, the more voters vote for A against B, the larger is P(A > B). Consider now the following case in which n=7 and the data of pairwise comparisons are the following:

	A	B	C
A	–	4,3	2,5
B	3,4	–	6,1
C	5,2	1,6	–

Here we have the majority cycle A > B > C > A.

Let $p_1 = P(A>B)$, $p_2 = P(B>C)$ and $p_3 = P(C>A)$. From the above additional assumption it follows that $1>p_2>p_3>½$. The probability that A is the right choice equals the probability that A is better than B and A is better than C. Denoting this probability by P(A), we in other words get:

$$P(A) = p_1 \cdot (1-p_3)$$

assuming that from the statements C > A and A > C one and only one is true. Similarly we get

$$P(B) = p_2 \cdot (1-p_1) \text{ and } P(C) = p_3 \cdot (1-p_2).$$

We notice that $(1-p_3) < (1-p_1)$ and that $p_2 > p_1$. Therefore, $P(A) < P(B)$. Consequently, the choice must be made between B and C. It is, however, easy to see that $P(A) > P(C)$. Thus, B's probability of being the right choice is largest and it should, therefore, be chosen.

This calculus does not always yield a unique result. Once more Condorcet resorts to the intuition and proposes the following general principle of choice which always yields the same end result as the above calculus whenever the latter is applicable: when there are three alternatives, eliminate the preference statement with the smallest supporting majority, and order the alternatives in the way that is consonant with the remaining two preference statements. In the previous example this principle results in the elimination of $A > B$. The remaining preference statements are $C > A$ and $B > C$. Hence the preference order is $B > C > A$. Again the argumentation is independent of the probability calculus of Condorcet.

From the view-point of the later development of the social choice theory perhaps the most interesting part of Condorcet's investigation concerns the case of arbitrarily many alternatives. Unfortunately his essay does not give a completely clear idea of his recommendation in this case. When we have at our disposal the results of pairwise comparisons Condorcet proposes the elimation of those statements which are backed by the smallest majorities. Proceeding in this fashion one will eventually end up with a non-cyclic preference order which then is the one sought for. Regrettably the description Condorcet gives leaves many details open. For example, it is not completely obvious which is the set from which one should eliminate the preference statements supported by smallest majorities (Black 1958, 169). As a matter of fact quite a few of the voting procedures proposed over the past two centuries can be seen as specifications of Condorcet's proposal. Due to the ambiguities of Condorcet's essay, however, these specifications are not equivalent with each other.

BORDA, CONDORCET AND THE MODERN GROUP CHOICE THEORY

The theory of group choice has been »invented» and »forgotten» many times during the period from the end of the 18th century to modern times (see Riker 1961). In Condorcet's and Borda's times the probability theorists were particularly interested in group choice theory. For example, Laplace, whose main contribution are in the probability theory proper, wrote on group choice problems. During the 19th century there was no accumulation of knowledge on the foundations of Borda's and Condorcet's works. Towards the end of the century C.L. Dodgson published some interesting pamphlets which, however, do not appear to be based on knowledge of the works of the two French masters. During the last century the esteem of Condorcet's works decreased because of the evaluations of some commentators who did not thoroughly understand the importance of the works. For example, Isaac Todhunter writes in his famous history book which was first published in 1865: »We must state at once that Condorcet's work is excessively difficult; the difficulty does not lie in the mathematical investigations, but in the expressions which are employed to introduce these investigations and to state their results: it is in many cases almost impossible to discover what Condorcet means to say. The obscurity and self contradiction are without any parallel, so far as our experience of mathematical works extends; some examples will be given in the course of our analysis, but no amount of examples can convey an adequate impression of the extent of these evils. We believe that the work has been very little studied, for we have not observed any recognition of the repulsive peculiarities by which it is so undesirably distinguished» (Todhunter 1949, 352). The main target of the criticism is Condorcet's probability calculus. Todhunter was not the only one who regarded Condorcet's essay as a treatise on probability theory. It was therefore to be expected that even those who otherwise took a favourable attitude towards his work in social philosophy simply ignored his study in social choice theory. Consequently this part of his production was simply forgotten. As a matter of fact it was not until after the Second World War that Condorcet's theory of voting procedures began to be viewed independently of his probability calculus. This reinterpretation was largely due to Black (1958). Condorcet himself would probably have been somewhat embarrashed at noticing that the winning criterion bearing his name is related to an example in which he had to abandon his probability calculus in favour of plain intuition. Nonetheless, the reinterpretation has undeniably increased Condorcet's academic prestige. On the other hand, the probability calculations and the underlying idea of »right» social choice are characteristic of the idea of the social science that Condorcet represented. Moreover, Condorcet's argumentation is applicable whenever there is a wider context with respect to which one can evaluate the »goodness» or »rightness» of decisions. Nevertheless, Condorcet's idea of social choice as a special case of the jury problem was not adopted by most of the modern social choice theorists. In the next section of this article we shall, however, encounter a method which could be viewed as an attempt to follow Condorcet's footsteps in the sense that the method is based on the assumption that the voters have opinions about the competence of each other in the choice of the right alternative or candidate.

If Condorcet's jury problem has been largely forgotten, his discussion of cyclical majorities has been in the focus of attention in the present day social choice theory. Indeed, the phenomenon of cyclical majorities is sometimes called the Condorcet paradox. This paradox has been one of the most central themes in the modern social choice research. Arrow's famous impossibility theorem deals with a procedure that would satisfy certain »reasonable» conditions and would produce a connected and transitive social preference order.[3] Hence, the requirements set upon the output aim explicitly at avoiding the Condorcet paradox.

Condorcet's discussion on cyclical majorities has provided motivation for even more recent research. Norman Schofield (1978; 1980) has employed the concepts of global, local and infinitesimal cycle set. These are specifications of the majority cycle concept that is the crux of the Condorcet paradox. Schofield's results on the topological properties of these sets specify the conditions of validity of some results in group choice theory, e.g. the theorems of McKelvey (1976; 1979). On the other hand, Schofield's results provide new solution concepts in the theory of voting games. A kind of application of Condorcet's ideas is the voting procedure proposed by Schwartz. In this procedure the Condorcet winner is chosen or alternatively (when there is no Condorcet winner) all the elements of such a majority cycle are chosen in which no element is defeated by some element outside the cycle (Schwartz 1972; Richelson 1978; Nurmi 1981 (b)). In this procedure one in effect admits that the problem of cyclical majorities is unsolvable and considers that there is a tie between all the alternatives in the majority cycle defined above. Hence, all the elements of the cycle should be chosen.

In the preceding we noticed that Condorcet gave a superficial characterization of his method in cases where there are more than three alternatives and the pairwise contests do not lead to a connected and transitive social preference order. It is quite sure, however, that Condorcet proposed a method which would be reducible to the method he outlined in extenso for three alternatives. During the past decades many methods have been designed that could be regarded as »specifications» of Condorcet's ideas. As was pointed out above the specifications are not, however, equivalent, but sometimes lead to different choice sets. One could envisage that when writing about the successive elimination of statements supported by smallest majorities, Condorcet meant either an elimination procedure taking into account the individual preference orders or a method based on pairwise comparisons. In the former case the elimination methods of Hare and Coombs would be suitable specifications. In the latter case, in turn, either minimax or maximin methods would be a suitable explication of Condorcet's line of reasoning. In

Hare's procedure one chooses the alternative that has been ranked first by a majority of voters, if there is one. If not, one eliminates that alternative which has the fewest first ranks among the alternatives. If after the elimination one finds an alternative regarded as best by the majority of voters (i.e. by more than 50 % of the voters), it is chosen. If one is still unable to find such an alternative, the elimination is continued until the majority is unanimous about the alternative ranked first (Straffin 1980; see also Nurmi 1981 (b)).

Another plausible explication of Condorcet's ideas is the voting procedure proposed by Coombs (Straffin 1980). Hare's procedure takes account of the alternatives ranked first by the voters, whereas Coombs' procedure pays attention to the alternatives ranked last. Actually Coombs' procedure chooses according to precisely the same criterion as Hare's procedure, but it eliminates successively those alternatives which have the largest number of last ranks in individual preference orders.

It is, however, unlikely that Condorcet would have been satisfied with these elimination procedures, for it can be shown that they do not necessarily choose the Condorcet winner when one exists (Nurmi 1981 (b)). Therefore, had Condorcet had the opportunity to investigate these methods, he would probably have stated that they are counterintuitive.

Maybe the maximin or minimax methods are closer to what Condorcet had in mind. These two methods do not differ much from each other. Therefore, it is sufficient to outline the maximin method only.[4] The input data of this method consist of the voting results from each pairwise comparison. Let the set of alternatives be $X = \{x_1, \ldots, x_k\}$. Then we denote by $\bar{s}_i$ the minimum support given to x_i when all pairwise comparisons are taken into account. Next we define $\bar{s}$ as follows: $\bar{s} = \max_i \bar{s}_i$. Finally the maximin set $\bar{X}_M$ is defined: $\bar{X}_M = \{x_j \in X \mid \bar{s}_j = \bar{s}\}$. Thus the maximin set consists of those alternatives that have the largest minimum support or that fare best when confronted with their toughest competitor. This set is always nonempty. Moreover, the Condorcet winner, whenever it exists, always belongs to the set $\bar{X}_M$ (Nurmi 1982). Viewed from this angle the maximin method could be the method that Condorcet had in mind.

Borda's memoir ended up with recommending a specific voting procedure. As was pointed out above, this procedure was indeed applied for some time in practice. After the initial success, however, the method was soon forgotten and remained that way until fairly recently. During the 1970's the method has been intensively studied. Thus, it has been given an axiomatic characterization (Young 1974). That is, the conditions both individually necessary and jointly sufficient for a method always to choose exactly the same alternatives as the Borda count, have been listed. It is clear that Borda's memoir has inspired also more generally the study of positional voting procedures (Young 1974; Young 1975).

The Borda count does not necessarily choose the Condorcet winner. Even though this is a characteristic of many voting procedures currently in use, it has, nonetheless, been regarded as such a crucial drawback that in the beginning of this century E.J. Nanson designed a modification of the Borda count that does not have this unpleasant property (see Black 1958, 186–188; Straffin 1980). Nanson's method consists actually of successive applications of the Borda count so that at each round the alternative having the smallest Borda score is eliminated and Borda count is again applied to the reduced alternative set. The procedure is continued until the required number of »best» alternatives is left.

Another method which tries to combine the virtues of Condorcet's and Borda's winning criteria, has been developed by Black (1958, 55–56). It simply chooses the Condorcet winner when one exists. Otherwise, the Borda winner is chosen. So the method combines both the uncontestability property of the Condorcet winner and the decisiveness property of the Borda count.

Focusing now on the general research strategies of Borda and Condorcet, one can observe that Borda's approach with its precise delineations and clear recommendations was undoubtedly the first analytic comparison of voting systems. This general approach has recently been pursued by e.g. Richelson (1979), Straffin (1980) and Niemi and Riker (1976) (see also Nurmi 1981 (a) and (b)). Condorcet's approach was more speculative. He tried to see the research problems in a wider context and compared various criteria of goodness. Condorcet's followers can be found in the research tradition originated by Arrow. The central foci of this tradition are various requirements of preference aggregation and the compatibility of the requirements. The best known representatives of this tradition are – in addition to Arrow (1963) – Sen (1970), Pattanaik (1971) and Plott (1976) (see also Mueller 1979 and Kelly 1978).

RATIONAL CONSENSUS PROCEDURE

An interesting way of combining the approaches (i) and (ii) discussed in the introduction is proposed by Lehrer and Wagner (1981). The authors outline a method for reaching a rational consensus in a group of persons concerning e.g. the probability of a hypothesis being true or the choice of therapy to be applied in the treatment of a patient. Surely these are not the kinds of issues one typically encounters in social choice settings, but Lehrer and Wagner argue that their method is also applicable in the more typical cases. The point of choosing these examples is that the method of reaching a rational consensus seems to fit most naturally the situations involving the aggregation or rather synthesis of different expert opinions. Let us take a look at what Lehrer and Wagner call the elementary method (in contradistinction to the extended model to be discussed briefly later on).

Consider a hypothesis H and a group N consisting of n members discussing the probability of H being true. Lehrer and Wagner start with the state called dialectical equilibrium, i.e. a state which is reached when in the course of the discussion about H, each member i of the group assigns a subjective probability p_i^O to H so that the continuation of the discussion does not change the opinion of any member of the group. We are now looking for a method to assign a probability to H in a way that would best summarise the information about H in the group, i.e. the method would represent the rational consensus about the probability of H.

In Lehrer's and Wagner's procedure each member i of N assigns a weight w_{ij} $(j=1, \ldots, n)$ to every member of N reflecting his/her opinion about the expertise or reliability of j in the issue at hand. A trivial way of assigning weights would be to give 1/n's weight to every member of N (including i). But obviously this need not to be the case, for i might well appreciate the expertise of some persons of the group more highly than that of others. Now the fact that i gives a positive weight to j's probability estimate obviously means that he/she wants to adjust his/her probability estimate on the basis of the information about j's estimate. A straight-forward way of adjustment is the weighted average of the probability assignments. Thus, denoting by p_i^O i's probability estimate at stage zero, we get

$$p_i^1 = w_{i1}p_i^O + w_{i2}p_2^O + \ldots + w_{in}p_n^O.$$

where $0 \leqslant w_{ij} \leqslant 1$ and $\sum_j w_{ij} = 1$.

Indeed, Lehrer and Wagner argue that there is a consistency argument in favour of this type of aggregation. That is, if i gives a nonzero weight to some j and yet refuses to aggregate his stage one probability in the above fashion, then he/she is in fact acting as if he/she would give a weight one to himself/herself and the weight zero to others including j. Hence, such a refusal to aggregate would be inconsistent.

Now for the whole group N the process of moving from stage zero to stage one can be represented as the matrix multiplication: WP, where

$$W = \begin{bmatrix} w_{11} & w_{12} \cdots w_{1n} \\ w_{21} & w_{22} \cdots w_{2n} \\ \cdots & \cdots \\ w_{n1} & w_{n2} \cdots w_{nn} \end{bmatrix} \quad \text{and } P = \begin{bmatrix} p_1^0 \\ p_2^0 \\ \vdots \\ p_n^0 \end{bmatrix}$$

From stage one we get the probability assignment vector of stage two by multiplying WP by W again:

$$W(WP) = W^2P.$$

Similarly, the probability vector of stage k is W^kP.

Now, under certain conditions the probability vector converges to the consensual probability vector in which all elements are identical. The following conditions would guarantee the convergence, i.e. be sufficient for it: (1) the condition of positive respect, that is, $w_{ij} \neq 0$, for all i and j, (2) the condition of weight constancy, that is, W remains the same throughout the procedure, and (3) the aggregation condition which states that the members of N are consistent in the sense discussed above.

The condition (1) is stringent, indeed. What it says is that every member should appreciate the judgement of every other member of N. Clearly when issues involve conflict of some kind, it is doubtful that this condition is satisfied. One can guarantee convergence with a less stringent condition, however. Let us say that i is reachable from j whenever there is a sequence of nonzero weights $w_{ik_1}, w_{k_1k_2}, \ldots, w_{k_sj}$, where $k_i \in N$ for all $i = 1, \ldots, s$. Now when every i is reachable from every j, we can say that there is a communication of respect in N. This property is much less stringent than (1) and yet guarantees the convergence when combined with (2) and (3).

In what Lehrer and Wagner call the extended model the conditions (2) and (3) are modified as well. Since these modifications do not seem to enhance the applicability of the procedure in social choice contexts, we shall not dwell on the extended model further. The definite virtue of Lehrer's and Wagner's elementary method is the fact that it can be implemented in a fairly straightforward way by simply asking the members of N to indicate their subjective probabilities and the weights. The rest can be performed by computers. Therefore, the fact that (2) can be replaced by a less stringent condition would only complicate the matters as one could anyway allow for weight modifications when new issues are voted upon.

But how to apply this method of reaching consensual probability to cases in which the choice is to be made between alternatives? The first possibility is based on the observation that the convergence to consensual probability vector occurs because there is a convergence of the weight matrix W^k as k increases. In the limit each of the rows of the weight matrix is identical. Now, this means that the individuals in the limit agree about the weight of each person in judging the issue at hand. A straight-forward way of applying the method to social choice would be to choose the person with the largest consensual weight to make the choice on behalf and with the authority of the group. This surely would be in accordance with the strategy (ii) discussed in the introduction. It is not, however, necessary that the voters would be committed to this method because in order to regard the method acceptable one has to approve *two* methods of aggregation: 1. the one which is used in computing the next stage probabilities of individuals, and 2. the one which determines the changes in the weight matrix over stages. Lehrer and Wagner build a strong case for 1., but it seems that 2. is more difficult to defend in a social choice context. The main objection to 2. is that the changes in the i'th row depend not only on i's possible »learning» (supposing that one resorts to the extended model and actually goes through the various stages allowing modifications of weights at each stage), but on other persons' weight assignments as well. There is some plausibility in the way in which the weights are modified. For example: $w_{ij}^1 = \sum_t w_{it} w_{tj}$, i.e. the new weights are obtained as weighted averages of the weights given by all persons. As weights in *this* averaging one uses the weights assigned to persons as experts of the issue at hand. Now it is questionable to presume that the same weights would be applicable in the averaging process involving the modification of weights and of probabilities. In other words, it is not self-evident that persons who would accept the 1. aggregation would also feel committed to the 2. aggregation.

In a social choice context this objection can be stated most emphatically if the above method of electing the person with the largest consensual weight is resorted to. However, what is at issue here is a general shortcoming of Lehrer's and Wagner's method as the consensual weights play a crucial role in other modifications of the method as well.

Another way of applying the method in social choice context is to utilise the utility values given by individuals to alternatives.

This is actually recommended by Lehrer and Wagner whenever the utilities are available. Consider again the set X of alternatives. Assume that the individuals assign utility values to the k alternatives. Hence we can construct the following matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} \cdots & a_{1k} \\ a_{21} & a_{22} \cdots & a_{2k} \\ \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} \cdots & a_{nk} \end{bmatrix}$$

Let the vector of consensual weights be the following:

$$C = \begin{bmatrix} w_1, \ldots, w_n \end{bmatrix}.$$

The consensual utility assignment would then be obtained by taking the matrix product CA. It would then be straight-forward to choose the socially most preferred elements: the alternative x_i is preferred to x_j if and only if the element on the i'th row in the CA matrix is larger than that of the j'th row. The most preferred alternative is, of course, the element which is assigned the largest consensual utility in the CA matrix.

Lehrer and Wagner are fully aware of the difficulties involved in utility measurement, especially as it turns out that this method requires that the unit of utility is the same for all individuals. Hence, the uniqueness of utility scales up to affine transformation is not sufficient for this method. Instead of dwelling on measurement problems, we shall, however, concentrate on other more specific issues related to Lehrer's and Wagner's method in social choice context. Three main objections will be made in addition to the criticism of the type 2. aggregation.

Firstly, the rational consensus method seems to be ill-suited to political decision making concerning e.g. public goods provision, because of its amenability to strategic misrepresentation of preferences. This is a common weakness of most voting methods, but in the case of Lehrer's and Wagner's method the problems of strategic behaviour are particularly marked. The reason is that, while it may be objected that the usual definition of manipulability (i.e. amenability to strategic misrepresentation of preferences) is counterintuitive in the sense that it presupposes that the voters know the preferences of each other, Lehrer's and Wagner's procedure in manipulable in a more straight-forward way. Let us assume that the variant of rational consensus method to be applied is the one in which the weight matrix only matters, i.e. the person with the largest consensual weight is chosen to make the decisions on behalf of the collectivity. We take an example from Lehrer and Wagner (1981, 29–32). Consider the following weight matrix:

0.5	0.49	0.009	0	0	0.001
0.1	0.8	0.1	0	0	0
0.009	0.49	0.5	0.001	0	0
0	0	0.001	0.5	0.49	0.009
0	0	0	0.05	0.9	0.05
0.001	0	0	0.009	0.49	0.5

Obviously there are two groups of mutual respect, viz. persons 1–3 and persons 4–6. There is a communication of respect between the groups (necessary for convergence) effected by members 1, 3, 4 and 6. Supposing that these are the true weights persons are giving to each other and themselves, the vector for consensual weights (reported in Lehrer's and Wagner's book) is: 0.053 0.262 0.053 0.053 0.524 0.053.

Obviously person 5 would be chosen as his/her consensual weight is by far the largest. Suppose now that person 2 slightly misrepresents his/her weight vector so as to give the following vector:

0.05 0.9 0.05 0 0 0

instead of the second row of the above matrix. This misrepresentation, ceteris paribus, leads to the following vector of consensual weights:

0.042 0.415 0.042 0.042 0.415 0.042

Now we have a tie between persons 2 and 5, clearly a preferred outcome from the view-point of person 2. All he/she had to do was to give a very small increment of weight to his/her own opinion. It could be conjectured that in general one would benefit from misrepresentation of weights so that either one's own weight or the weight of the most respected member in one's subgroup is increased. Of course, it is no proof of the conjecture that it seems to be valid in one example. Nonetheless, the example shows both that the method is manipulable in the ordinary sense and that it can in this case be manipulated in an obvious way without having to know exactly the preferences of all the others.

Actually the above manipulation of weight assignments is a new possibility for strategic behaviour additional to the manipulability of utility assignments. If one knows the consensual weight matrix and the utility assignment of others, there are certainly cases in which one can benefit from not revealing one's true utilities, e.g. in cases where there would be a tie between two alternatives neither one of which is the best from the persons view-point. In that case the person in question would benefit from assigning the better alternative of the two largest utility value available thereby, ceteris paribus, guaranteeing the choice of that alternative.

Secondly, the rational consensus method is sensitive to very small variations in the weight assignments. This is, in my view, a serious flaw as it affects the very justifiability of the method also in cases where genuinely honest and sincere people get together to reach a consensus. Let us consider again one of Lehrer's and Wagner's weight matrices (Lehrer & Wagner 1981, 31):

0.5	0.49	0.01	0	0	0
0.0495	0.9	0.0495	0	0.001	0
0.01	0.49	0.5	0	0	0
0	0	0	0.5	0.49	0.01
0.001	0	0	0.0495	0.9	0.0495
0	0	0	0.01	0.49	0.5

The consensual weight vector is again

0.042 0.415 0.042 0.042 0.415 0.042.

So far the procedure works nicely in the sense that a small »perturbation» of the matrix considered earlier – small changes in all the rows of the earlier matrix – does not change the consensual weight vector. But if person 5 changes his/her weight assignment into the following vector:

0.002 0 0 0.049 0.9 0.049,

the result is a dramatic change in consensual weights:

0.057 0.554 0.056 0.028 0.277 0.028.

Now person 2 beats the others hands down. And yet the only change required for this overwhelming victory of person 2 is the assignment of weight 0.002 instead of 0.001 to person 1 by person 5 and the corresponding reduction of weights of persons 4 and 6. Now, could one really presuppose that people in general are able to make so accurate weight assignments as to be able to say whether a person's weight should be 0.001 instead of 0.002? I don't think so. If I'm right, then the consensual weight vector would seem to contain a touch of arbitrariness not easily reconcilable with the idea of rational consensus.

Thirdly, Lehrer's and Wagner's method is not monotonic in preferences. In other words, it can happen that while x_i is the alternative with the largest consensual utility given C and A matrices, it may no more be the social choice when A changes into A' where the only difference is that x_i's utility value is increased in some person's opinion. Of course, when this kind of phenomenon occurs, there must have been a change in C as well. All the same, the fact that a person's preference for an alternative increases while the other person's preferences remain the same, provides no guarantee that the likelihood of the alternative's being chosen also increases.

In spite of the above criticism of Lehrer's and Wagner's method in social choice context, it is only fair to say that the work continues the tradition of Condorcet in a very important sense which has largely been overlooked by other social choice theorists. Ian White (1979, 117–118) points out that modern social choice theorists are concerned with aggregating the preferences of all those persons affected by the decision to be made. They operate within the context of political democracy. But there are collective decisions in which the very fact that a person is affected by the decision constitutes a reason for excluding him/her from the decision making. The case in point is judicial decision making. Condorcet's work was related specifically to the kinds of problems involving truth, falsity and probability of statements like verdicts. As we argued above, this line of reasoning has not survived in social choice theory. In Lehrer's and Wagner's work we now have a worthy successor of Condorcet's interesting and important approach.

NOTES

1 Riker would probably argue that the distinction between (i) and (ii) is the same as the one that differentiates populism from liberalism in democratic systems. See Riker (1982).

2 Borda used this example. See DeGrazia (1953). The article contains an English translation of Borda's memoir. The preferences are written in the usual way, i.e. the uppermost alternative is the one regarded the best by the voter etc.

3. See Arrow (1963). Of course, the studies of Black (1958) should be mentioned in this context.

4. The minimax-set and the process converging to it have been studied by Kramer (1977). See also the earlier study of Simpson (1969).

REFERENCES

Arrow, Kenneth J. — *Social Choice and Individual Values,* 2. ed. New York: Wiley 1963.

Baker, Keith M. — »Condorcet, Marquis de» in *The Encyclopedia of Philosophy,* ed. P. Edwards, New York: The Macmillan Company 1967, Vol. 2, pp. 182–184.

Black, Duncan — *The Theory of Committees and Elections*, Cambridge: Cambridge University Press 1958.

Condorcet — *Essai sur l'application de l'analyse à la probabilité des décisions rendues à la pluralité des voix,* Paris 1785.

DeGrazia, Alfred — »Mathematical Derivation of an Election System», *Isis* 44, 1953, pp. 42–51.

Granger, Gilles-Gaston — *La Mathematique Sociale du Marquis de Condorcet*, Paris: Presses Universitaires de France 1956.

Kelly, Jerry S. — *Arrow Impossibility Theorems*, New York: Academic Press 1978.

Kramer, Gerald H. — »A Dynamical Model of Political Equilibrium», *Journal of Economic Theory* 16, 1977, pp. 310–334.

Lehrer, Keith & Wagner, Carl — *Rational Consensus in Science and Society*, Dordrecht: D. Reidel 1981.

McKelvey, Richard D. — »Intransitives in Multidimensional Voting Models and Some Implications for Agenda Control», *Journal of Economic Theory* 12, 1976, pp. 472–482.

McKelvey, Richard D. — »General Conditions for Global Intransitivities in Formal Voting Models», *Econometrica* 47, 1979, pp. 1085–1112.

Mueller, Dennis C. — *Public Choice*, Cambridge: Cambridge University Press 1979.

Niemi, Richard G. & Riker, William H. — »The Choice of Voting Systems», *Scientific American* 234, 1976, pp. 21–27.

Nurmi, Hannu — »*On Taking Preferences Seriously*», manuscript 1982.

Nurmi, Hannu — »On the Properties of Voting Systems», *Scandinavian Political Studies* 4, 1981 (a), pp. 19–32.

Nurmi, Hannu — »Voting Procedures», *British Journal of Political Science* (to appear in 1983) 1981 (b).

Pattanaik, Prasanta K. — *Voting and Collective Choice*, Cambridge: Cambridge University Press 1971.

Plott, Charles R. — »Axiomatic Social Choice Theory», *American Journal of Political Science* 20, 1976, pp. 511–596.

Richelson, Jeffrey T. — »A Comparative Analysis of Social Choice Functions, II», *Behavioral Science* 23, 1978, pp. 38–44.

Richelson, Jeffrey T. — »A Comparative Analysis of Social Choice Functions: I, II, III: A Summary», *Behavioral Science* 24, 1979, p. 355.

Riker, William H. — »Voting and the Summation of Preferences», *American Political Science Review* 55, 1961, pp. 900–911.

Riker, William H. — *Liberalism against Populism*, San Francisco: W.H. Freeman 1982.

Schofield, Norman — »General Relevance of the Impossibility Theorem in Dynamical Social Choice», Annual Meeting of the American Political Science Association, New York 1978.

Schofield, Norman — »Formal Political Theory», *Quality & Quantity* 14, 1980, pp. 249–275.

Schwartz, Thomas — »Rationality and the Myth of the Maximum», *Nous* 6, 1972, pp. 97–117.

Sen, Amartya K. — *Collective Choice and Social Welfare*, San Francisco: Holden-Day 1970.

Simpson, Paul B. — »On Defining Areas of Voter Choice», *The Quarterly Journal of Economics* LXXXIII, 1969, pp. 478–490.

Straffin, Philip D., Jr — *Topics in the Theory of Voting*, Boston: Birkhäuser 1980.

Todhunter, Isaac — *A History of the Mathematical Theory of Probability from the Time of Pascal to That of Laplace*, New York: Chelsea Publishing Company 1949 (original 1865).

Young, H. Peyton — »An Axiomatization of Borda's Rule«, *Journal of Economic Theory* 9, 1974, pp. 43–52.

Young, H. Peyton — »Social Choice Scoring Functions«, *SIAM Journal of Applied Mathematics* 28, 1975, pp. 824–838.

White, Ian — »Condorcet: Politics and Reason« in *Philosophers of the Enlightenment*, ed. S.C. Brown, Brighton: The Harvester Press 1979.